

# ANALYTIC TORSION, VORTICES AND POSITIVE RICCI CURVATURE

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**ABSTRACT.** We characterize the global maximizers of a certain non-local functional defined on the space of all positively curved metrics on an ample line bundle  $L$  over a Kähler manifold  $X$ . This functional is an adjoint version, introduced by Berndtsson, of Donaldson's  $L$ -functional and generalizes the Ding-Tian functional whose critical points are Kähler-Einstein metrics of positive Ricci curvature. Applications to (1) analytic torsions on Fano manifolds (2) Chern-Simons-Higgs vortices on tori and (3) Kähler geometry are given. In particular, proofs of conjectures of (1) Gillet-Soulé and Fang (concerning the regularized determinant of Dolbeault Laplacians on the two-sphere) (2) Tarantello and (3) Aubin (concerning Moser-Trudinger type inequalities) in these three settings are obtained. New proofs of some results in Kähler geometry are also obtained, including a lower bound on Mabuchi's  $K$ -energy and the uniqueness result for Kähler-Einstein metrics on Fano manifolds of Bando-Mabuchi. This paper is a substantially extended version of the preprint [10] which it supersedes.

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## 1. INTRODUCTION

**1.1. Background.** Consider the two-dimensional sphere  $S^2$  equipped with its standard Riemannian metric  $g_0$  of constant positive curvature, normalized so that the corresponding volume form  $\omega_0$  gives unit volume to  $S^2$ . A celebrated inequality of Moser-Trudinger-Onofri proved in its sharp form by Onofri [56], asserts that

$$(1.1) \quad \log \int_{S^2} e^{-u} \omega_0 \leq - \int_{S^2} u \omega_0 + \frac{1}{4} \int_{S^2} du \wedge d^c u$$

for any, say smooth, function  $u$  on  $S^2$ , where the last term is the  $L^2$ -norm of the gradient of  $u$  in the conformally invariant notation of section 1.3 below.

As is well-known the inequality above has a rich geometric content and appears in a number of seemingly unrelated contexts ranging from the problem of prescribing the Gauss curvature in a conformal class of metrics on  $S^2$  (the Yamabe and Nirenberg problems [24]) to sharp critical *Sobolev inequalities* [7] and lower bounds on *free energy functionals* in mathematical physics [56, 58, 23]. There is also a *spectral* interpretation of the Moser-Trudinger-Onofri inequality [56, 57]: the “determinant”  $\det \Delta_g$ , of the Laplacian on  $S^2$  seen as a functional on space of all unit area conformal metrics  $g$  on  $S^2$ , is maximized precisely on the standard round metric - up to conformal automorphisms. More precisely,  $\det \Delta_g$  is defined using the zeta function regularization of the product of all strictly positive eigenvalues and its maximizers are hence the metrics of the form  $e^{-u}g_0$  satisfying the the constant positive curvature equation:

$$\omega_0 + dd^c u = e^{-u}\omega_0,$$

where  $dd^c u = \Delta_{g_0} u \omega_0 / 4\pi$  (using the notation in section 1.3). The bridge between this spectral problem and the inequality 1.1 is given by the *Polyakov anomaly formula* [24], which first appeared in Physics in the path integral (random surface) approach to the quantization of the bosonic string.

One of the main motivations for the present paper comes from a conjecture of Gillet-Soulé concerning the determinant of the Dolbeault Laplacian acting on the space of  $(0, q)$ -forms of a vector bundle over a Kähler manifold. In the one-dimensional case it can be formulated as follows:

**Conjecture.** (*Gillet-Soulé*). *Let  $(X, \omega_0)$  be a complex curve with a fixed Hermitian metric. The determinant of the Dolbeault Laplacian  $\Delta_{\bar{\partial}}$  considered as a functional on the space of all smooth Hermitian metrics on a holomorphic line bundle  $L \rightarrow X$  is bounded from above.*

The conjecture was motivated by Arakelov geometry, in particular the arithmetic Riemann-Roch theorem and is equivalent to the boundedness (from below) of certain arithmetic Betti-numbers ([42]; see also [41] p. 526-527). In the case of  $S^2$  the conjecture was confirmed by Fang [39], who by symmetrization reduced the problem to the case when the metric on  $L$  is invariant under rotation around an axes of  $S^2$ , earlier treated by Gillet-Soulé [42]. Fang also put forward the following more precise form of the conjecture above, when  $L$  is ample:

**Conjecture.** (*Fang*). *In the case of  $(S^2, \omega_0)$  equipped with the standard round metric  $\omega_0$  the upper bound in the previous conjecture is achieved precisely for the Fubini-Study metric on  $L$  (up to scaling).*

In other words the maximizers are conjectured to be precisely the metrics on the line bundle  $L$  whose curvature form is constant in the sense

that it is given by the invariant metric  $\omega_0$  (assuming that  $L$  has positive degree). Fang was also motivated by the similarities between the conjecture above and a classical inequality of Szegő for Toeplitz operators on the unit-circle. It should also be pointed out that the conjecture above implies, by well-known arguments in Arakelov geometry involving Minkowski's theorem for lattices, an effective arithmetic Riemann-Roch theorem on the projective line [41] (for a precise statement see the preprint [10], section 3.5).

**1.2. The present paper.** In this paper the positive solution of Fang's conjecture will be deduced from a general result about the maximizers of a certain non-local functional  $\mathcal{F}_{\omega_0}$  defined on the space of all positively curved Hermitian metrics on an ample line bundle  $L$  over a Kähler manifold  $(X, \omega_0)$ . In fact, a more precise inequality than the one conjectured by Fang will be obtained (Corollary 3) which implies both Fang's conjecture *and* the Moser-Trudinger-Onofri inequality (and hence the extremal properties of  $\det \Delta_g$ , as well). The inequality obtained for  $L = \mathcal{O}(m)$ , i.e. the degree  $m$  line bundle on  $S^2$ , is equivalent to the following upper bound,

$$(1.2) \quad \log\left(\frac{\det \Delta_{\bar{\partial}_u}}{\det \Delta_{\bar{\partial}_0}}\right) \leq -\frac{1}{2}\left(\frac{1}{(m+2)}\right) \int du \wedge d^c u (\leq 0),$$

where we have expressed a metric  $h$  on  $\mathcal{O}(m)$  as  $h = e^{-u} h_0^{\otimes m}$  in terms of the Fubini-Study metric on  $S^2$ , which clearly implies Fang's conjecture above. The extremals in the first inequality above will also be characterized.

Generalizations in two different directions will also be considered. In the case of a higher dimensional Fano manifold  $X$  (i.e  $X$  admits a metric of positive Ricci curvature) upper bounds on the *analytic torsions* associated to powers of the anti-canonical line bundle  $-K_X$  will be given. Compared to the higher dimensional version of the conjecture of Gillet-Soulé this amounts to bounds on *alternating* products of determinants of Dolbeault Laplacians in *all* degrees  $(0, q)$ . Interestingly, it will appear that while it is crucial to assume the positivity of the curvature of the metrics on  $L$  in higher dimensions this is not so when  $X$  is a complex curve. In the case of complex curves of higher genus we will prove uniqueness for mean field type equations with one vortex on complex curves, partly confirming a conjecture of Tarantello [64, 65] in the context of self-dual Chern-Simons-Higgs equations.

The functional  $\mathcal{F}_{\omega_0}$  referred to above is an “adjoint version”, introduced by Berndtsson, of Donaldson's (normalized)  $L$ -functional and generalizes the Ding-Tian functional whose critical points are Kähler-Einstein metrics. In particular, new proofs of some results in Kähler geometry are also obtained, including a lower bound on Mabuchi's  $K$ -energy and the uniqueness result for Kähler-Einstein metrics on Fano manifolds of Bando-Mabuchi (see section 1.7 for precise references).

Before turning to the precise statement of the main result we will first introduce the general setup.

**1.3. Setup and statement of the main results.** Let  $L \rightarrow X$  be a holomorphic line bundle over a compact complex manifold  $X$  of complex dimension  $n$ . Denote by  $\text{Aut}_0(X, L)$  the group of automorphism of  $(X, L)$  in the connected component of the identity, modulo the elements covering the identity on  $X$ . The line bundle  $L$  will be assumed *ample*, i.e. there exists a Kähler form  $\omega_0$  in the first Chern class  $c_1(L)$  and a “weight”  $\psi_0$  on  $L$  such that  $\omega_0$  is the normalized curvature  $(1, 1)$ –form of the hermitian metric on  $L$  locally represented as  $h_0 = e^{-\psi_0}$ . In this notation, the space of all positively curved smooth hermitian metrics on  $L$  may be identified with the open convex subset

$$\mathcal{H}_{\omega_0} := \{u : \omega_u := dd^c u + \omega_0 > 0\}$$

of  $\mathcal{C}^\infty(X)$ , where  $d^c := i(-\partial + \bar{\partial})/4\pi$ , so that  $dd^c = \frac{i}{2\pi}\partial\bar{\partial}$ . Hence,  $\omega_u$  is the normalized curvature form of the metric  $e^{-u}h_0$  on  $L$  representing the first Chern class of  $L$  in  $H^2(X, \mathbb{Z})$ . The natural multiplicative action of  $\mathbb{R}^*$  on metrics on  $L$  hence corresponds to an *additive* action of  $\mathbb{R}$  on  $\mathcal{H}_{\omega_0}$ , that we will sometimes refer to as “scaling”. Note that the natural action of  $\text{Aut}_0(X, L)$  on the space of all metrics on  $L$  corresponds to the action  $(u, F) \mapsto v := F^*(\psi_0 + u) - \psi_0$  so that, in particular,  $\omega_v = F^*\omega_u$ . Occasionally, we will also work with the closure  $\overline{\mathcal{H}}_{\omega_0}$  of  $\mathcal{H}_{\omega_0}$  in  $L^1(X, \omega_0)$ , coinciding with the space of all  $\omega_0$ –plurisubharmonic functions on  $X$ , i.e. the space of all upper semi-continuous functions  $u$  which are absolutely integrable and such that  $\omega_u \geq 0$  as a  $(1, 1)$ –current.

We equip the  $N$ –dimensional complex vector space  $H^0(X, L + K_X)$  of all holomorphic sections of the adjoint bundle  $L + K_X$  where  $K_X$  is the canonical line bundle on  $X$ , with the Hermitian product induced by  $\psi_0$ , i.e.

$$\langle s, s \rangle_{\psi_0} := i^{n^2} \int_X s \wedge \bar{s} e^{-\psi_0},$$

identifying  $s$  with a holomorphic  $n$ –form with values in  $L$ . We will use additive notation for tensor products of line bundles.

Next, we will introduce the two functionals on  $\mathcal{H}_{\omega_0}$  which will play a leading role in the following. First, consider the following energy functional

$$(1.3) \quad \mathcal{E}_{\omega_0}(u) := \frac{1}{(n+1)!V} \sum_{i=1}^n \int_X u (dd^c u + \omega_0)^i \wedge (\omega_0)^{n-i},$$

where  $V := \text{Vol}(\omega_0)$  is the volume of  $L$ , which seems to first have appeared in the work of Mabuchi [54] and Aubin [4] in Kähler geometry ( $\mathcal{E}_{\omega_0} = -F_{\omega_0}^0$  in the notation of [70]). It also appears in Arithmetic (Arakelov) geometry as the top degree component of the secondary Bott-Chern class of  $L$  attached to the Chern character.

The second functional  $\mathcal{L}_{\omega_0}$  may be geometrically defined as  $\frac{1}{N}$  times the logarithm of the quotient of the volumes of the unit-balls in  $H^0(X, L +$

$K_X$ ) defined by the Hermitian products induced by the metrics  $\psi_0$  and  $\psi_0 + u$  [11]. Concretely, this means that

$$(1.4) \quad \mathcal{L}_{\omega_0}(u) := -\frac{1}{N} \log \det(\langle s_i, s_j \rangle_{\psi_0+u}),$$

where  $1 \leq i, j \leq N$  and  $s_i$  is any given base in  $H^0(X, L + K_X)$  which is orthogonal wrt  $\langle \cdot, \cdot \rangle_{\psi_0}$ . The functional  $\mathcal{L}_{\omega_0}(u)$  may also be invariantly expressed as a *Toeplitz determinant*:

$$(1.5) \quad \mathcal{L}_{\omega_0}(u) := -\frac{1}{N} \log \det(T[e^{-u}]),$$

where  $T[e^{-u}]$  is the *Toeplitz operator with symbol  $e^{-u}$*  (compare formula 5.2 in the appendix). If  $N = 0$  we let  $\mathcal{L}_{\omega_0}(u) := -\infty$ . The normalizations are made so that the functional

$$\mathcal{F}_{\omega_0} := \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$$

is invariant under the action of  $\mathbb{R}$  and hence descends to a functional on the space of all Kähler metrics in  $c_1(L)$ . An element  $u$  in  $\mathcal{H}_{\omega_0}$  will be said to be *critical* (wrt  $L + K_X$ ) if it is a critical point of the functional  $\mathcal{F}_{\omega_0}$  on  $\mathcal{H}_{\omega_0}$ , i.e. if  $u$  is a smooth solution in  $\mathcal{H}_{\omega_0}$  of the Euler-Lagrange equations  $(d\mathcal{F}_{\omega_0})_u = 0$ . These equations may be written as the highly non-linear Monge-Ampère equation:

$$(1.6) \quad \frac{1}{Vn!} (dd^c u + \omega_0)^n = \beta(u),$$

where  $\beta(u)$  is the Bergman measure associated to  $u$  (formula 2.5 below). This latter measure depends on  $u$  in a *non-local* manner and is strictly positive precisely when  $L + K_X$  is *globally generated*, i.e. when there, given any point  $x$  in  $X$ , exists an element  $s$  in  $H^0(X, L + K_X)$  such that  $s(x) \neq 0$ . For example, since  $L$  is ample, this condition holds when  $L$  is replaced by  $kL$  for  $k$  sufficiently large.

By definition, a critical point  $u$  is a priori only a *local* extremum of  $\mathcal{F}_{\omega_0}$ . But the next theorem relates *global* maximizers of  $\mathcal{F}_{\omega_0}$  and its critical points:

**Theorem 1.** Let  $L$  be an ample line bundle over a compact complex manifold  $X$ . Then the absolute maximum of the functional  $\mathcal{F}_{\omega_0}$  on  $\mathcal{H}_{\omega_0}$ , defined above, is attained at any critical point  $u$ . Moreover if the adjoint line bundle  $L + K_X$  is globally generated any smooth maximizer of  $\mathcal{F}_{\omega_0}$  on  $\overline{\mathcal{H}}_{\omega_0}$  is unique (up to addition of constants) modulo the action of  $\text{Aut}_0(X, L)$ . In particular, such a maximizer is critical.

In the case when the ample line bundle  $L = -K_X$ , so that  $X$  is a Fano manifold, the space  $H^0(X, L + K_X)$  is one-dimensional and hence  $\mathcal{L}_{\omega_0}(u) = -\frac{1}{N} \log \int e^{-(u+\psi_0)}$ . Then it is well-known that any critical point may be identified with a Kähler-Einstein metric on  $X$ , i.e. the corresponding Kähler metric satisfies

$$\text{Ric } \omega_u = \omega_u$$

It should be emphasized that the *existence* of critical points of  $\mathcal{F}_{\omega_0}$  is a difficult issue. For example, in the case  $L = -K_X$  fundamental conjectures of Yau, Tian, Donaldson [70, 34, 66] relate the existence problem to a notion of algebro-geometric stability. In section 4.4 some conjectures concerning the case of a general line bundle  $L$  are proposed. As explained in remark 21 there is a weak solution (in the sense of pluripotential theory) to the critical point equation if the functional  $\mathcal{F}_{\omega_0}$  is *proper* or *coercive* in a suitable sense. In the case of  $L = -K_X$  this amounts to saying that  $X$  is *analytically stable* in the sense of Tian [70]. Moreover, by Proposition 23 and the subsequent remark the property of admitting a critical point is an open condition in the moduli space of polarized manifolds  $(X, L)$  as long as  $\text{Aut}_0(X, L)$  is trivial.

However, in the presense of large symmetry groups existence of critical points can be readily established. Hence, we next assume that  $(X, L)$  is *K-homogenous*, i.e. that  $X$  admits a transitive action by a compact semi-simple Lie group  $K$ , whose action on  $X$  lifts to  $L$ . We will then take  $\omega_0$  as the unique Kähler form in  $c_1(L)$  which is invariant under the action of  $K$  on  $X$ .

**Corollary 2.** *Let  $L \rightarrow X$  be a  $K$ -homogenous ample holomorphic line bundle over a compact complex manifold  $X$  and denote by  $\omega_0$  be the unique  $K$ -invariant Kähler metric in  $c_1(L)$ . Then, for any function  $u$  in  $\mathcal{H}_{\omega_0}$*

$$-\mathcal{L}_{\omega_0}(u) \leq -\mathcal{E}_{\omega_0}(u)$$

*with equality iff the function  $u$  is constant, modulo the action of  $\text{Aut}_0(X, L)$ .*

Surprisingly, specializing to the case when  $X$  is a complex curve (i.e.  $n = 1$ ) allows one to take  $u$  as *any* smooth function (which is *not* true in higher dimensions, as shown in [50] in the case when  $L = -K_X$ ; see remark 16). More generally, we can then take  $u$  to be in the Sobolev space  $W^{2,1}(X)$  of all functions  $u$  on  $X$  such that  $u$  and its differential  $du$  are square integrable. We will first consider the homegenous case in one dimension, i.e.  $X = \mathbb{P}^1$ , the complex projective line (i.e. topologically  $X = S^2$ , the two-sphere) and hence  $K = SU(2)$ . Identifying  $S^2$  with the one-point compactification of  $\mathbb{C}_z$  (compare section 3.3.1) we then get the following corollary

**Corollary 3.** *Let  $u$  be a function in the Sobolev space  $W^{2,1}(S^2)$  on the two-sphere  $S^2$  and denote by  $\omega_0$  the volume form corresponding to the standard invariant metric on  $S^2$  of unit-area. Then*

$$\log \det(c_{ij}) \int_{\mathbb{C}} \frac{z^i \bar{z}^j}{(1 + z\bar{z})^m} e^{-u} \omega_0 \leq -(m+1) \int_{S^2} u \omega_0 + \left(\frac{m+1}{m+2}\right) \frac{1}{2} \int_{S^2} du \wedge du^c$$

*where  $1/c_{ij} = (m+1) \binom{m}{i} \binom{m}{j}$  with equality iff there exists a Möbius transformation  $F$  of  $S^2$  such that  $\omega_u = F^* \omega_0$ .*

The case when  $m = 0$  gives a new proof of the Moser-Trudinger-Onofri inequality 1.1. The reduction of the proof of Corollary 3 to Corollary 2

is based on properties of a projection operator  $P_{\omega_0}$  (formula 1.11). As pointed out by Fang [39] the determinant in the previous corollary may also be expressed as the integral of  $\sum_i e^{-u(x_i)}$  over the  $N$  fold product of  $S^2$  (with  $N = m + 1$ ) of the  $SU(2)$ -invariant probability measure with density

$$\rho_N(x_1, \dots, x_N) := \prod_{1 \leq i < j \leq N} \|x_i - x_j\|^2 / Z_N$$

where  $1/Z_N = N^N \binom{N-1}{0} \dots \binom{N-1}{N-1} / N!$ , written in terms of the ambient Euclidian norm in  $\mathbb{R}^3$ . In the physics literature this integral appears as the free energy for the Gibbs ensemble of a *Coulomb gas* of unit-charge particles (i.e one component plasma) confined to the sphere [22] in a neutralizing uniform background. In the case of a general line bundle  $\mathcal{L}_{\omega_0}$  may be expressed in terms of a determinantal random point process (see section 5.1)

In the non-homogenous one dimensional case, i.e. higher genus curves the assumption in Theorem 1 about global generation of  $L + K_X$  turns out to be superfluous. A simple application of the Riemann-Roch theorem shows that  $L + K_X$  is not globally generated precisely when  $L := L_p$  is an effective degree one line bundle, i.e.  $L$  has a holomorphic section  $s$  whose zero divisor is an irreducible point  $p$  in  $X$ .

**Theorem 4.** *Let  $L$  be a degree one line bundle over a complex curve  $X$  of genus  $g \geq 1$ . Then the functional  $\mathcal{F}_{\omega_0}$  on  $\mathcal{H}_{\omega_0}$ , admits a unique (modulo scaling) critical point  $u$  on  $\mathcal{C}^\infty(X)$ . Moreover  $u$  is a maximizer of  $\mathcal{F}_{\omega_0}$  on  $\mathcal{C}^\infty(X)$ .*

The *existence* of a critical point in the previous theorem follows from a variational argument, using the general Moser-Trudinger inequality of Fontana [40]. This variational approach is closely related to the one used by Troyanov [72] in his study of constant curvature metrics on Riemann surfaces with conical singularities.

**1.4. Applications to twisted analytic torsion.** Given any line bundle  $L$  over a complex curve  $(X, \omega_0)$  equipped with a Kähler metric  $\omega_0$  of constant curvature we will denote by  $\det \Delta_{\bar{\partial}}$  the functional on the space of all smooth Hermitian metrics on  $L$  which to an Hermitian metric associates the zeta-regularized product of the strictly positive eigenvalues of the corresponding  $\bar{\partial}$ -Laplacian (see section 2.4). Combining Corollary 3 above with the anomaly formula of Bismut-Gillet-Soulé [16] now implies the following positive solution of Fang's conjecture

**Corollary 5.** *For any line bundle over  $L$  over  $\mathbb{P}^1$  equipped with its invariant metric the functional  $\det \Delta_{\bar{\partial}}$  has a unique maximizer (modulo scaling), namely the Hermitian metric on  $L$  whose curvature form  $\omega$  has constant curvature.*

In fact, the proof of the previous corollary will give the stronger statement that the inequality 1.2 for  $\det \Delta_{\bar{\partial}_u}$  stated in the introduction holds and that this latter inequality is equivalent to Corollary 3. Note that a

direct consequence of the previous corollary is the following response to a variant of Kac's classical question "Can one hear the shape of a drum?" [47]: if the  $\bar{\partial}$ -Laplacian on *some* power  $\mathcal{O}(m)$  induced by a smooth metric on  $\mathcal{O}(1) \rightarrow \mathbb{P}^1$  has the same spectrum (including multiplicities) as the  $\bar{\partial}$ -Laplacian induced by the standard  $SU(2)$  invariant metric then the two metrics coincide up to scaling.

In the case when  $X$  is a genus one curve, we obtain the following

**Corollary 6.** *Let  $L$  be degree one line bundle over a complex curve  $X$  of genus one and write  $L = L_p$  for a unique point  $p$  on  $X$ . Then the functional  $\det \Delta_{\bar{\partial}}$  has a unique maximizer (modulo scaling), namely the Hermitian metric on  $L_p$  whose curvature form is the constant curvature metric  $\omega_p$  with a conical singularity at  $p$  (see below:  $\omega_p$  satisfies equation (ii)' in 1.10 for  $t = 1$ ).*

A natural higher dimensional generalization of the regularized determinant associated to a line bundle  $L$  over a Kähler manifold  $(X, \omega_0)$  (we will assume that  $\omega_0 \in c_1(L)$ ) is the *analytic torsion* (see section 2.4). The following theorem can be seen as a generalization of a weak form of Fang's result on  $S^2$  to certain Fano manifolds. It is formulated in terms of the following invariant  $R(X)$  of a Fano manifold  $X$  :

$$(1.7) \quad R(X) = \sup_{\omega_t} \{t : \text{Ric } \omega_t > t\omega_t\} > 0$$

with  $\omega_t$  ranging over all Kähler metrics in  $c_1(-K_X)$  (this invariant was studied very recently by Székelyhidi, [63]; the strict positivity follows from [4]).

**Theorem 7.** *Let  $X$  be a Fano manifold with  $R(X) > 1 - 1/n^2$ . Then, for  $k$  sufficiently large, the analytic torsion associated to  $-kK_X$  is bounded from above when considered as a functional on the space of all positively curved metrics on  $-kK_X$ . In particular, the boundeness holds for any Fano surface. Moreover, when  $(X, \omega_0)$  is equal to  $\mathbb{P}^2$  equipped with the standard invariant metric  $\omega_0$  the upper bound holds for any positive  $k$  and the maximum is achieved precisely on the metrics of constant curvature  $\omega_0$ , i.e. precisely on the Fubini-Study metric on  $-kK_X$  (up to scaling).*

The fact that the condition on  $R(X)$  above is satisfied on Fano surfaces (i.e. on the Del Pezzo surfaces) follows from [68, 52, 63]. If  $X$  admits a Kähler-Einstein metric then  $R(X) = 1$  (but the converse is not true) [63]. It would be interesting to know to what extent the condition on  $R(X)$  above is satisfied in higher dimension? One motivation for the previous theorem comes from the following observation (see remark 20): after a suitable scaling the analytic torsion  $\mathcal{T}_k(u)$  for a metric  $h_{ku}$  on  $-kK_X$  (expressed in terms of  $u \in \mathcal{H}_{\omega_0}$  as  $h_{ku} = e^{-ku} h_0^{\otimes k}$ ) has the following asymptotics due to Bismut-Vasserot [17]:

$$\lim_{k \rightarrow \infty} \mathcal{T}_k(u) = -\mathcal{S}(\mu_u, \mu_0)$$



i.e. minus the *relative entropy* of the probability measure  $\mu_u := \omega_u^n / Vn!$  with respect to the measure  $\mu_0$ . It is a well-known fact that  $\mathcal{S}(\mu, \nu) \geq 0$  with equality if and only if the two measures coincide. Hence, the previous theorem can be seen as an analogue of this latter fact for *finite*  $k$ . We will also show that, in the case of  $(\mathbb{P}^n, \omega_0)$  for  $n \leq 2$  the analytic torsion functionals  $\mathcal{T}_k$  are geodesically *concave* on  $\mathcal{H}_{\omega_0}$  equipped with its symmetric space metric. This should be compared with a recent result of von Renesse-Sturm [60] saying that on a Riemannian manifold  $(X, g_0)$  of unit-volume the relative entropy  $\mathcal{S}(\mu, dVol_{g_0})$  is convex with respect to the  $L^2$ -Wasserstein metric on the space of all probability measures  $\mu$  on  $X$  precisely when  $g_0$  has semi-positive Ricci curvature.

**1.5. Application to vortices and metrics with conical singularities.** Let  $(X, \omega_0)$  be a Riemann surface with a Kähler (i.e. area) form  $\omega_0$  of unit area. Given a parameter  $\mu \in \mathbb{R}_+$  and a fixed point  $p$  the corresponding (normalized) *mean field equation* with a single vortex at  $p$  is defined as

$$(1.8) \quad (i) \Delta u = \eta(e^{g_p - u} - 1)$$

where  $\Delta$  is the Laplacian with respect  $\omega_0$  and  $g_p$  is the corresponding Green function with a pole at  $p$  (note that any solution satisfies  $\int_X e^{g_p - u} = 1/\eta$ ) A similar mean field type equation was also studied very recently in [49]:

$$(1.9) \quad (ii) \Delta w + \eta e^w = \eta \delta_p$$

which is equivalent to the previous one when  $\eta = 4\pi$  (just set  $w = g_p - u$ ). These two equations appear in the mean field limit of a statistical mechanics model for vortices in fluids [58, 23]. For a very recent account of mean field equations and their relations to self-dual *Chern-Simons-Higgs equations* see for example [65] (see also section 4.5). Equivalently, setting  $t := \eta/4\pi$  the pseudo-metrics  $\Omega_t := \omega_{u/t}$  and  $\omega_t := \omega_{g_p - w/t}$  satisfy

$$(1.10) \quad (i') \text{Ric } \Omega_t := t\Omega_t - [p] + (1-t)\omega_0, \quad (ii') \text{Ric } \omega_t := t(\omega_t - [p])$$

For  $t = 1$  both equations hence describe a metric  $\omega_t$  of unit area with unit positive Gauss curvature and a conical singularity at  $p$  of angle  $4\pi$ . The following theorem should be seen in the light of the well-known fact that conical constant curvature metrics may be non-unique for angles strictly larger than  $2\pi$  (for the case of the two-sphere see [73, 38]).

**Theorem 8.** *Let  $X$  be a Riemann surface of genus one with a marked point  $p$  and let  $\omega_0$  be the standard invariant metric on  $X$  (i.e.  $(X, p, \omega_0)$  may be identified with  $(\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}), 0, \frac{i}{2}dz \wedge d\bar{z})$  for  $\tau$  in the upper half plane). Then the mean field equations 1.8 and 1.9 both admit a unique solution when  $\eta \in ]0, 4\pi + \epsilon]$  for some  $\epsilon > 0$ .*

The *existence* of a solution for  $\eta \in ]0, 8\pi[$  is well-known and follows from Moser-Trudinger inequalities and basic variational arguments as above. Uniqueness in the case when  $\eta \in ]0, 8\pi[$  was conjectured by Tarantello [64]

(see also the discussion on page 158 in [65]). The uniqueness for  $\mu = 4\pi$  was established very recently in [49] (Theorem 3.2) using function theory on the elliptic curve  $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ , where the uniqueness of solutions for equation 1.9 was conjectured for  $\eta \in [4\pi, 8\pi[$  as well. Under the latter assumption on  $\eta$  the authors proved the uniqueness of solutions invariant under the natural  $\mathbb{Z}_2$  action  $z \mapsto -z$  on  $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ . Remarkably, it was also shown in [49] that uniqueness of equation 1.9 fails for  $\eta = 8\pi$  as soon as there is some solution (which was shown to happen when  $\tau = e^{i2\pi/3}$ ). For uniqueness results in the different setting of plane domains see [6, 23]. As a byproduct of the present proof we will also obtain uniqueness of the *linearized* equations (in the case of  $\eta = 4\pi$  this “linear stability” was proved in [49] under the assumption of  $\mathbb{Z}_2$  symmetry).

It would be interesting to know whether the present methods could be complemented in order to deal with the uniqueness problem when  $\eta \in ]4\pi, 8\pi[$  and the case when  $X$  is a Riemann surface of genus at least two. In the latter case the present proof gives a generalization of the previous corollary to a variant of the mean field equation where the factor  $e^{g_p - u}$  is multiplied with a certain smooth and strictly positive function  $f_u$  (see equation 3.13). For a fixed function  $f$  (often independent of  $u$ ) such generalized mean field equations have been studied extensively in the literature (see [65] and references therein).

Finally, it should be emphasized that it is the positive sign of the parameter  $\mu$  that is the source of the delicate analytical properties of the equation 1.8. Indeed, when  $\eta < 0$  it follows readily from the maximum principle that 1.8 has a unique solution. In this case the equations are equivalent to the abelian *Yang-Mills-Higgs equations* (see section 4.5).

**1.6. Applications to Kähler geometry.** These applications are formulated and proved in section 4.

**1.7. Further relations to previous results.** The relation between the Moser-Trudinger type inequalities and Kähler-Einstein metrics of positive curvature in higher dimensions seems to first have been suggested by Aubin [4] who also proposed a conjecture that we will be proved in section 4.2. In the case when  $L = -K_X$  the first statement of Theorem 1 is a result of Ding-Tian[31] and the “uniqueness” of critical points (i.e. Kähler-Einstein metrics in this case) was proved earlier by Bando-Mabuchi [5]. See [12] for a generalization of this latter result to functions of “finite energy”, in the case when  $\text{Aut}_0(X, L)$  is trivial (compare remark 22).

The extremal property of the critical points in Theorem 1 can also be seen as an analog of a result of Donaldson (Theorem 2 in [35]) who furthermore assumed that  $\text{Aut}_0(X, L)$  is discrete. In this latter setting the role of the space  $H^0(X, L + K_X)$  is played by  $H^0(X, L)$  equipped with the scalar products induced by the weight  $\psi_0 + u$  and the integration measure  $(\omega_u)^n/n!$  Note however that in Donaldson’s setting the functional

$\tilde{\mathcal{F}}$  corresponding to  $\mathcal{F}_{\omega_0}$  is *minimized* on its critical points (compare section 4.3 and the discussion in section 5 in [15]). In fact, when  $X$  is an arithmetic variety the functional  $\tilde{\mathcal{F}}$  seems to first have appeared in the work of Bost [20] and Zhang [75]. In the terminology of [35] these latter critical points correspond to *balanced metrics*. Donaldson used his result, combined with the deep convergence results in [34] for balanced metrics, in the limit when  $L$  is replaced by a large tensor power, to prove a lower bound on Mabuchi's K-energy functional (under the assumption that  $\text{Aut}_0(X, L)$  be trivial). It will be shown in section 4 how to deduce this latter result more directly from Theorem 1 above. After the first version of the present paper (the preprint ref) appeared another proof by Li ref of the lower bound on Mabuchi's K-energy functional appeared which does not use that  $\text{Aut}_0(X, L)$  be trivial either.

It should also be pointed out that the inequality proved by Donaldson corresponds to a *lower* bound on  $\mathcal{F}_{\omega_0}(u)$  in the present setting, which however will depend on  $u$  through its volume form  $(\omega_u)^n/n!$  (see the end of section 4.3 ).

Note also that Rubinstein [51, 50] recently gave a different complex geometric proof of the Moser-Trudinger-Onofri inequality 1.1 using the inverse Ricci operator and its relation to various energy functionals in Kähler geometry. See also Müller-Wendland [53] for a proof of the result on extremals of determinants of the scalar Laplacian using the Ricci flow. It should also be pointed out that extremals related to twisted analytic torsion on Kähler manifolds were also studied in [53]. However the torsion was twisted by the Quillen metric on a certain virtual vector bundle earlier introduced by Donaldson in the seminal work [32].

## 1.8. Outline of the proofs of the main results.

*Theorem 1.* The starting point is the recent work [15] of Berndtsson which shows that  $\mathcal{F}_{\omega_0}$  is concave along geodesics with respect to the Riemann metric on the space  $\mathcal{H}_{\omega_0}$  introduced by Mabuchi [53]. Moreover, the concavity is *strict* modulo the action of  $\text{Aut}(X, L)_0$ . Hence, if  $\mathcal{H}_{\omega_0}$  were geodesically convex (i.e. if any two points in  $\mathcal{H}_{\omega_0}$  could be connected by a smooth geodesic) Theorem 1 would follow immediately. However, since there are no existence results for such geodesic segments (apart from the toric setting) one of the key points of the present paper is to show how to systematically use the global pluripotential theory developed in [11, 13] to get around this difficulty. This is achieved by working with continuous “geodesics” in the  $L^1$ -closure  $\overline{\mathcal{H}}_{\omega_0}$  of  $\mathcal{H}_{\omega_0}$ . A priori such a geodesic may hence touch the “boundary”  $\overline{\mathcal{H}}_{\omega_0} - \mathcal{H}_{\omega_0}$ , but we will exclude this scenario by showing that  $\mathcal{F}_{\omega_0}$  has no maximizers on the “boundary” of  $\mathcal{H}_{\omega_0}$  (if  $K_X + L$  is globally generated). Following [11, 12] this crucial latter fact is shown by extending  $\mathcal{F}_{\omega_0}$  to a (Gâteaux) differentiable function on all of  $C^0(X)$ , by replacing  $\mathcal{E}_{\omega_0}$  with the composed map  $\mathcal{E}_{\omega_0} \circ P_{\omega_0}$ , where  $P_{\omega_0}$  is the following (non-linear) projection operator from  $C^0(X)$  onto

$$\mathcal{C}^0(X) \cap \overline{\mathcal{H}}_{\omega_0} :$$

$$(1.11) \quad P_{\omega_0}[u](x) = \sup \{v(x) : v \in \mathcal{H}_{\omega_0}, v \leq u\}$$

Somewhat remarkably, this latter projection operator is also used to show that any critical point of  $\mathcal{F}_{\omega_0}$  is automatically a global maximizer on all of  $C^\infty(X)$  when  $n = 1$ . Moreover, it is shown that the assumption that  $K_X + L$  be globally generated is not needed when  $n = 1$  by refining the arguments in [15].

*The applications to analytic torsion.* These applications are based on the fact that the analytic torsion functional may be expressed as  $\mathcal{F}_{\omega_0}$  plus an explicit curvature term (which vanishes when  $X$  is a torus). Moreover, the main contribution in this term (when  $L$  is replaced by a large tensor power) comes from the Ricci curvature for  $\omega_0$  and this leading term appears with a useful negative sign when the Ricci curvature is positive.

*The applications to vortices.* These are obtained by studying a deformation of  $\mathcal{F}_{\omega_0}$ , depending on  $\mu$ , which preserves the concavity when  $\mu \in ]0, 4\pi]$ . The main new difficulty comes from the fact that one has to work with forms  $\omega_u$  degenerating at the fixed point  $p$ .

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**Organization.** In section 2 preliminaries for the proofs of the main results appearing in section 3 are given. The proof of the uniqueness statement in the main theorem relies on higher order regularity for “geodesics” defined by inhomogeneous Monge-Ampère equations. An alternative proof based on considerably more elementary regularity results is given in section 3.7, as well extensions to degenerate boundary data in the one-dimensional case. Finally, in section 4 the limit when the line bundle  $L$  is replaced by a large tensor power is studied and a new proof of the lower bound on Mabuchi’s  $K$ -energy for a polarized projective manifold. The proof of Theorem 7 concerning upper bounds on twisted analytic torsions on Fano manifolds and a proof of a conjecture of Aubin are also given and relations to Donaldson’s work are discussed. Some conjectures are also proposed. In the appendix some formulas involving Bergman kernels are recalled and a “Bergman kernel proof” of Theorem 12 is given, as well as generalizations to a degenerate setting in the one-dimensional case.

## 2. PRELIMINARIES: GEODESICS AND FUNCTIONALS

**2.1. Geodesics and psh paths.** The infinite dimensional space  $\mathcal{H}_\omega$  inherits an *affine* Riemannian structure from its natural imbedding as an open set in  $\mathcal{C}^\infty(X)$ . Mabuchi, Semmes and Donaldson (see [25] and references therein) introduced another Riemannian structure on  $\mathcal{H}_\omega$  (modulo the constants) defined in the following way. Identifying the tangent space of  $\mathcal{H}_\omega$  at the point  $u$  with  $\mathcal{C}^0(X)$  the squared norm of a tangent vector  $v$  at the point  $u$  is defined as

$$\int_X v^2 (\omega_u)^n / n!.$$

In fact, this metric is induced by a natural symmetric space structure on  $\mathcal{H}_\omega$ . However, the *existence* of a geodesic  $u_t$  in  $\mathcal{H}_\omega$  connecting any given points  $u_0$  and  $u_1$  is an open and perhaps even dubious problem. There are two problems: it is not known if *i*)  $u_t$  smooth, *ii*)  $\omega_{u_t}$  is strictly positive, as a current. As is well-known such a geodesic may, if it exists, be obtained as the solution of a homogenous Monge-Ampère equation (see below). In the following we will simply take this characterization as the *definition* of a geodesic. It will also be important to consider the larger space  $\overline{\mathcal{H}}_{\omega_0} \cap C^0(X)$ , since a priori the path  $u_t$  may leave  $\mathcal{H}_\omega$ .

**Definition 9.** A continuous path in  $\overline{\mathcal{H}}_{\omega_0} \cap C^0(X)$   $u_t$  will be called a  $\mathcal{C}^0$ -geodesic connecting  $u_0$  and  $u_1$  if  $U(w, x) := u_t(x)$ , where  $t = \log |w|$ , is continuous on

$$M := \{1 \leq |w| \leq e\} \times X := A \times X$$

with  $dd^c U + \pi_X^* \omega_0 \geq 0$  and

$$(2.1) \quad (dd^c U + \pi_X^* \omega_0)^{n+1} = 0$$

in the interior of  $M$  in the sense of pluripotential theory [43, 29], where  $\pi_X$  denotes the projection from  $M$  to  $X$ .

As shown in [13, 12]  $U(w, x)$  exists and is uniquely defined as the extension from  $\partial M$  obtained as the upper envelope

$$(2.2) \quad U(w, x) = \sup \{V(w, x) : V \in \mathcal{H}_{\pi_X^* \omega_0}(M), V \leq U \text{ on } \partial M\},$$

where  $\mathcal{H}_{\pi_X^* \omega_0}(M)$  denotes the set of all smooth functions  $V$  on  $M$  such that  $dd^c U + \pi_X^* \omega_0 > 0$ . If  $u_t$  is such that  $dd^c U + \pi_X^* \omega_0 \geq 0$  then  $u_t$  will be called a *psh path* (or a *subgeodesic*). In local computations we will often make the identification  $u_t(x) = U(w, x)$  extending  $t$  to a complex variable in a strip in the complex plane. Then  $u_t(x)$  is independent of the imaginary part of  $t$  and is hence *convex* wrt real  $t$ . In this notation a binomial expansion in equation 2.1 shows the equation for a smooth geodesic  $u_t$  may be written as

$$\partial_{\bar{t}} \partial_t u_t \omega_{u_t}^n / n! - \partial_X \partial_{\bar{t}} u \wedge \bar{\partial}_X \partial_t u \wedge \omega_{u_t}^{n-1} / (n-1)! = 0,$$

where the subscript  $X$  indicate derivatives along  $X$  and  $\partial_t$  and  $\partial_{\bar{t}}$  denote the holomorphic and anti-holomorphic partial derivatives wrt  $t$ , i.e.

$$(2.3) \quad c(u_t) := \partial_{\bar{t}}\partial_t u_t - |(\partial(\partial_{\bar{t}}u)|_{\omega_{u_t}}|^2 = 0$$

In the proof of the uniqueness part of Theorem 1 we will have great use for the following regularity result for geodesics in  $\overline{\mathcal{H}}_{\omega_0}$ , shown by Chen [25]. See also [19] for a detailed analysis of the proof and some refinements. The proof uses the method of continuity combined with very precise a priori estimates on the perturbed Monge-Ampère equations.

**Theorem 10.** *(Chen) Assume that the boundary data in the Dirichlet problem 2.1 for the Monge-Ampère operator on  $M$  is smooth on  $\partial M$ . Then  $U \in \mathcal{C}_{\mathbb{C}}^{1,1}(M)$ . More precisely, the mixed second order complex derivatives of  $U$  are uniformly bounded, i.e. there is a positive constant  $C$  such that*

$$0 \leq (dd^c U + \pi_X^* \omega_0) \leq C(\pi_X^* \omega_0 + \pi_A^* \omega_A)$$

where  $\omega_A$  is the Euclidian metric on  $A$ .

In the statement above we have used the (non-standard) notation  $\mathcal{C}_{\mathbb{C}}^{1,1}(M)$  for the set of all functions  $U$  such that, locally, the current  $dd^c U$  has coefficients in  $L^\infty$ . Such a  $U$  is called *almost  $\mathcal{C}^{1,1}$*  in [19]. Note that if  $U \in \mathcal{H}_{\pi_X^* \omega_0}(M)$  then this is equivalent to  $U$  having a bounded Laplacian  $\Delta_M U$ , where  $\Delta_M$  is the Laplacian on  $M$  wrt the Kähler metric  $\pi_X^* \omega_0 + \pi_A^* \omega_A$  on  $M$ . As will be explained in section 3.7 the proof of the uniqueness statement in Theorem 1 may actually be obtained by only using the bounds on the derivatives of  $u_t$  on  $X$  for  $t$  fixed. As shown very recently in [13] such bounds may be obtained by working directly with the envelope 2.2.

**Theorem 11.** *Assume that the boundary data in the Dirichlet problem 2.1 for the Monge-Ampère operator on  $M$  is in  $\mathcal{C}^{1,1}(\partial M)$ . Then  $u_t \in \mathcal{C}_{\mathbb{C}}^{1,1}(X)$ . More precisely, the mixed second order complex derivatives of  $u_t$  on  $X$  are uniformly bounded, i.e. there is a positive constant  $C$  such that*

$$0 \leq (dd^c u_t + \omega_0) \leq C\omega_0$$

on  $X$ .

One of the virtues of this latter approach is that the proof is remarkably simple when  $X$  is homogenous.

**2.2. The functional  $\mathcal{L}_{\omega_0}$ .** First note that the functional  $\mathcal{L}_{\omega_0}(u)$  defined by formula 1.5 is increasing on  $\mathcal{C}^0(X)$ , wrt the usual order relation. This is an immediate consequence of the basic geometric interpretation in [11] of  $\mathcal{L}_{\omega_0}(u)$  as proportional to the logarithmic volume of the unit-ball in the Hilbert space  $H^0(X, L + K_X)$  equipped with the Hermitian product induced by the weight  $\psi_0 + u$ . Alternatively, this fact follows from formula 2.4 below which shows that the differential of the functional  $\mathcal{L}_{\omega_0}$  on  $\mathcal{C}^0(X)$  may be represented by the positive measure  $\beta_u$ . Integrating  $\beta_u$  along a

line segment in  $\mathcal{C}^0(X)$  equipped with its affine structure then shows that  $\mathcal{L}_{\omega_0}(u)$  is increasing.

The *differential* of the functional  $\mathcal{L}_{\omega_0}$  on  $\mathcal{C}^0(X)$  is given by

$$(2.4) \quad (d\mathcal{L}_{\omega_0})_u = \beta_u,$$

in the sense that given any smooth function  $v$  we have that

$$d(\mathcal{L}_{\omega_0}(u + tv))/dt|_{t=0} = \int_X \beta_u v,$$

where  $\beta_u$  is the *Bergman measure associated to  $u$* . This latter measure is the positive measure on  $X$  defined as

$$(2.5) \quad \beta_u = (i^{n^2} \frac{1}{N} \sum_{i=1}^N s_i \wedge \bar{s}_i e^{-\psi_0}) e^{-u}$$

in terms of any given orthonormal base  $(s_i)$  in the Hilbert space  $H^0(X, L + K_X)$  equipped with the Hermitian product induced by the weight  $\psi_0 + u$  (compare section 5.1). In particular this means that  $\beta_u$  may be represented as  $e^{-u}$  times a strictly positive smooth measure on  $X$  if  $L + K_X$  is globally generated. The proof of formula 2.4 follows more or less directly from the definition (see [15] for a geometric argument).

The following theorem, which is direct consequence of a result of Berndtsson about the curvature of direct image bundles [15], considers the *second* derivatives of  $\mathcal{L}_{\omega_0}$  along a psh path. As a courtesy to the reader a new proof of the theorem, using Bergman kernels, is given in the appendix.

**Theorem 12.** (*Berndtsson*) *Let  $u_t$  be a continuous psh path in  $\mathcal{H}_{\omega_0}$ . Then the function  $t \mapsto \mathcal{L}_{\omega_0}(u_t)$  is convex. Moreover, if  $\mathcal{L}_{\omega_0}(u_t)$  is affine and  $u_t$  is a smooth psh path with  $\omega_{u_t} > 0$  on  $X$  for all  $t$ , then there is an automorphism  $S_1$  of  $(X, L)$ , homotopic to the identity, such that  $u_1 - u_0 = S_1^* \psi_0 - \psi_0$ .*

The convexity statement in [15] assumed in fact that  $u_t$  be *smooth*. However, by uniform approximation the convexity statement above in fact holds for any *continuous* psh path in  $\mathcal{C}^0(X)$ . Indeed, if  $u_t$  is such a path, then there exists, for example by Richberg's approximation theorem [28], a sequence  $U^j$  converging uniformly towards  $U$  on  $M$  such that  $dd^c U^j + \pi^* \omega_0 > 0$ . Applying the theorem above to each  $U^j$  and letting  $j$  tend to infinity then gives that  $f(t) := \mathcal{L}_{\omega_0}(u_t)$  is a uniform limit of convex functions and hence convex, proving the claim.

However, for the statement concerning the affine properties of  $\mathcal{L}_{\omega_0}(u_t)$  the argument in [15] requires that  $\omega_{u_t}$  be reasonably smooth in  $(t, x)$  (essentially  $\mathcal{C}^3$ -smooth). Moreover, the assumption that  $\omega_t > 0$  is crucial to be able to define the vector fields  $V_t$  that integrate to the automorphism  $S_1$  (see formula 2.6 below).

2.2.1. *The vector field  $V_t$  and the strict convexity of  $\mathcal{L}_{\omega_0}$ .* Given a psh path  $u_t$  such that  $u_t$  and  $\partial_t u$  (the partial holomorphic derivative wrt  $t$ ) are smooth on  $X$  with  $\omega_{u_t} > 0$  we let  $V_t$  be the family of vector fields on  $X$  of type  $(1, 0)$  defined by the equation

$$(2.6) \quad \omega_{u_t}(V_t, \cdot) = \bar{\partial}_X(\partial_t u),$$

where  $\bar{\partial}_X$  is the  $\bar{\partial}$ -operator on  $X$ . As shown in [15] if the functional  $\mathcal{L}(u_t)$  is affine wrt  $t$  then the vector field  $V_t$  has to be holomorphic on  $X$  for each  $t$ . Moreover, it then follows that  $V_t$  is holomorphic wrt  $t$  as well (this uses that  $u_t$  is automatically a geodesic; a slight variant of this argument is recalled in section 3.7). Integrating  $V_t$  will hence give rise to the automorphism  $S_1$  in Theorem 12. As explained above a major difficulty is that, for a given geodesic  $u_t$ , it is not known whether  $\omega_{u_t} > 0$ , nor if it is smooth on  $X \times ]0, 1[$  and the main part of the proof of the uniqueness statement in Theorem 1 will consist in showing that these conditions *are* satisfied for a geodesic connecting two critical points of  $\mathcal{F}_{\omega_0}$ . In the proof of Theorem 4 we will also have to handle the case when  $\omega_{u_t}$  vanishes at a finite number of points. To this end we will prove a generalization of Berndtsson's theorem in the one-dimensional case in section 5.3 in the appendix.

It was not explicitly pointed out in [15] that  $V_t$  defined above lifts to the total space of  $L$ , but this fact follows from Lemma 12 in [34]. For completeness and future reference we next give the proof of this lifting property emphasizing that it holds in a non-compact setting as well:

**Lemma 13.** *Let  $Y$  be a (possibly non-compact) complex manifold and  $\pi : L \rightarrow X$  a holomorphic line bundle equipped with a metric  $h_0$  with strictly positive normalized curvature form  $\omega_0$ . If  $V$  is a holomorphic vector field of type  $(1, 0)$  on  $X$  such that*

$$\omega_0(V, \cdot) = \bar{\partial}_X v$$

*for some complex valued smooth function  $f$  on  $Y$ , then  $V$  lifts to a holomorphic vector field on the total space of  $L$ .*

*Proof.* Following [35] the lifted vector field  $\tilde{V}$  may be explicitly defined by

$$\tilde{V} := V_{hor} - vW,$$

where  $V_{hor}$  is the horisontell lift to  $L$  of  $V$  (wrt the Chern connection induced by  $h_0$ ) and  $W$  is the generator of the natural  $\mathbb{C}^*$  action along the fibers of  $L$ . To verify that  $\tilde{V}$  is indeed holomorphic fix an arbitrary point  $p$  in  $Y$  and a trivialization  $s$  of  $L$  over a neighbourhood of  $p$  such that  $|s|_{h_0}^2 = e^{-\phi(z)}$ . Take holomorphic coordinates  $z$  centered at  $p$ . Then the trivialization of  $L$  induce local holomorphic coordinates  $(z, w)$  on  $L$  such that the various objects above may be expressed as follows [28]:

$$W = w \frac{\partial}{\partial w}, \quad V = \phi^{i\bar{j}} v_{\bar{i}} \frac{\partial}{\partial z_j} \quad A = \frac{dw}{w} - \phi_j dz_j$$



where  $A$  is the one-form  $A$  on  $L$  induced by the Chern connection (using the usual index notation and summation convention for partial derivatives wrt  $z$  and where  $\phi^{i\bar{j}}$  is the matrix representing the inverse of the matrix  $\phi_{i\bar{j}}$ ). Now by definition  $d\pi_*(V_{hor}) = V$  and  $A[V_{hor}] = 0$ , which in view of the above relations forces

$$V_{hor} = V + W\phi^{i\bar{j}}v_{\bar{i}}\phi_j = V + W\langle\partial\phi, V\rangle$$

Since by assumption  $V$  is holomorphic  $\bar{\partial}V_{hor} = W(\bar{\partial}(\phi^{i\bar{j}}v_{\bar{i}}\phi_j))$  and hence, using  $\bar{\partial}V = 0$  again gives

$$(\omega^{i\bar{j}}v_{\bar{i}}\phi_j)_{\bar{k}} = (\phi^{i\bar{j}}v_{\bar{i}})\phi_{j\bar{k}} = \phi^{i\bar{j}}v_{\bar{i}}\phi_{j\bar{k}} = v_{\bar{k}},$$

i.e.  $\bar{\partial}V_{hor} = W(\bar{\partial}v)$ . All in all this means that  $\bar{\partial}\tilde{V} = 0$ .  $\square$

**2.3. Energy functionals.** The functional  $\mathcal{E}_{\omega_0}$  defined in the introduction by formula 1.3 may be alternatively defined by the following variational property: the differential of the functional  $\mathcal{E}_{\omega_0}$  on  $\mathcal{H}_{\omega_0}$  is given by the following formula:

$$(2.7) \quad (d\mathcal{E}_{\omega_0})_u = \omega_u^n/n!$$

$\mathcal{E}_{\omega_0}$  may also be expressed in terms of a generalized Dirichlet type energy  $J_{\omega_0}$ :

$$(2.8) \quad -\mathcal{E}_{\omega_0}(u) = \frac{1}{V}(J_{\omega_0}(u) - \int u\omega_0)$$

where  $J_{\omega_0}$  is Aubin's energy functional

$$(2.9) \quad J_{\omega_0}(u) := \sum_{i=1}^{n-1} \frac{i+1}{(n+1)!} \int du \wedge du^c \wedge (\omega_0)^i \wedge (\omega_u)^{n-1-i}$$

which is clearly non-negative on  $\mathcal{H}_{\omega_0}$  (the definition differs from that in [70] p. 58 by a factor of  $Vn!$ ). Note that if  $n = 1$  then

$$J_{\omega_0} := \frac{1}{2} \int du \wedge du^c$$

is the classical Dirichlet energy on a Riemann surface, which is independent of  $\omega_0$  and non-negative for *any*  $u$ . Occasionally, we will also use the functional

$$(2.10) \quad I_{\omega_0}(u) = \int_X u(\omega_0^n - \omega_u^n)/n!,$$

which is also non-negative on  $\mathcal{H}_{\omega_0}$ . In fact, the following well-known inequality holds

$$(2.11) \quad I - J \geq J/n,$$

which is proved by a simple integration by parts argument.

The following generalization from [11] of formula 2.7 to the functional  $\mathcal{E}_{\omega_0} \circ P_{\omega_0}$ , where  $P_{\omega_0}$  is the non-linear projection 1.11, will be crucial for the proof of Theorem 1:

**Theorem 14.** *The functional  $\mathcal{E}_{\omega_0} \circ P_{\omega_0}$  is Gâteaux differentiable on  $\mathcal{C}^0(X)$ . Its differential at the point  $u$  is represented by the measure  $\omega_{P_{\omega_0}u}^n/n!$ , i.e. given  $u, v \in \mathcal{C}^0(X)$  the function  $\mathcal{E}_{\omega_0}P_{\omega_0}(u + tv)$  is differentiable on  $\mathbb{R}_t$  and*

$$(2.12) \quad d\mathcal{E}_{\omega_0}P_{\omega_0}(u + tv)/dt_{t=0} = \int_X v \omega_{P_{\omega_0}u}^n/n!$$

Note that for a merely continuous function  $u$  the functional  $\mathcal{E}_{\omega_0} \circ P_{\omega_0}$  may be defined in the sense of pluripotential theory, but in the present case  $P_{\omega_0}u$  is actually  $C^{1,1}$ –smooth [8] and hence  $\mathcal{E}_{\omega_0} \circ P_{\omega_0}$  is defined as an integral of a density in  $L_{loc}^\infty$ . As for the second derivatives of  $\mathcal{E}_{\omega_0}$  we have the following Proposition which is well-known (at least in the smooth case):

**Proposition 15.** *The following properties of  $\mathcal{E}_{\omega_0}$  and  $J_{\omega_0}$  hold:*

- The functional  $\mathcal{E}_{\omega_0}$  on  $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^0(X)$  is concave wrt the affine structure on  $\mathcal{C}^0(X)$ .
- Let  $u_t$  be a  $\mathcal{C}^0$ - geodesic in  $\overline{\mathcal{H}}_{\omega_0}$  connecting  $u_0$  and  $u_1$ . Then the functional  $t \mapsto \mathcal{E}_{\omega_0}(u_t)$  is affine and continuous on  $[0, 1]$ , while  $J_{\omega_0}(u_t)$  is convex.

*Proof.* (A proof also appears in [12]). Recall the following well-known formula (see for example [11]):

$$(2.13) \quad d_t d_t^c \mathcal{E}_{\omega_0}(u_t) = t_*(dd^c U + \pi^* \omega_0)^{n+1}/(n+1)!,$$

where  $t_*$  denotes the natural push-forward map from  $M$  to  $\mathbb{C}_t$ . In particular, setting  $u_t = u_0 + tu$  gives for real  $t$   $d^2 \mathcal{E}_{\omega_0}(u_t)/d^2 t = - \int_X |\partial u_t|^2 \omega_0^n \leq 0$  (compare formula 2.3) which proves the first point of the proposition when  $u$  is smooth. To handle the general case one takes  $u_j$  in  $\mathcal{H}_{\omega_0}$  converging uniformly to  $u$  and uses that, according to Bedford-Taylor’s classical results,  $\mathcal{E}_{\omega_0}$  is continuous under uniform limits in  $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^0(X)$  (see also [11]). This shows that  $\mathcal{E}_{\omega_0}(u_t)$  is the limit of concave functions and hence concave. To prove the last point take a sequence  $U^j$  converging uniformly to  $U$  on  $M$  and such that  $dd^c U^j + \pi^* \omega_0 > 0$  (compare the discussion below Theorem 12). By Bedford-Taylor  $(dd^c U^j + \pi^* \omega_0)^{n+1}$  tends weakly to  $(dd^c U + \pi^* \omega_0)^{n+1}$  in the interior of  $M$ . Hence, formula 2.13 shows that the second real derivatives of  $\mathcal{E}_j(t) := \mathcal{E}(u_t^j)$  tend weakly to zero in the sense of distributions for  $t \in ]0, 1[$ . But since the sequence  $\mathcal{E}_j(t)$  of smooth concave functions tends to  $\mathcal{E}(t)$  it follows that  $\mathcal{E}(t)$  is affine on  $]0, 1[$ . Finally, to see that continuity up to the boundary holds one proceeds as follows: since  $U$  is continuous on the compact set  $M$  the family  $u_t$  tends to  $u_0$  and  $u_1$  uniformly when  $t \rightarrow 0$  and  $t \rightarrow 1$ , respectively. Finally, since  $\mathcal{E}$  is continuous under uniform limits in  $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^0(X)$  this proves that  $\mathcal{E}$  is continuous up to the boundary on  $[0, 1]$ . Finally, the convexity of  $J_{\omega_0}(u_t)$  now follows immediately from formula 2.8 and the fact that  $u_t$  is convex in  $t$ .  $\square$

*Remark 16.* The natural condition to obtain non-negativity of the functional  $J_{\omega_0}$  when  $n > 1$  is that  $\omega_u \geq 0$ . On the other hand as shown in [50] (Lemma 2.1), there are examples of general smooth  $u$  with  $J_{\omega_0} < 0$  for any manifold  $X$  of dimension  $n > 1$ . For a completeness we give a simple proof of this fact when  $n = 2$  which is valid for the functional  $I_{\omega_0} - J_{\omega_0}$  as well: both these functionals may be decomposed as

$$A \int (-u)(ddu^c)^2 + B \int (-u)(ddu^c) \wedge \omega_0,$$

where  $A$  and  $B$  are strictly positive constants. Now take  $u$  such that the first term above is non-zero. Replacing  $u$  with  $-u$  if necessary we may even assume that the first term is strictly *negative*. Finally, replacing  $u$  with a sufficiently large positive multiple  $tu$  we see that the whole expression is negative (more precisely the first term is of the order  $t^3$ , while the second term is of the order  $t^2$ ) proving the fact alluded to above. As a direct consequence it was shown in [50], in the case  $L = -K_X$ , that any such function  $u$  violates the inequality in Theorem 1. A similar argument applies to any homogeneous line bundle  $L$  as in Corollary 2. Indeed, without affecting the value of  $J_{\omega_0}(u)$  we may assume that  $\int_X u \omega_0^n = 0$  so that  $-\mathcal{E}_{\omega_0}(u) = J_{\omega_0}(u) < 0$ . Now, using the notation explained in the beginning of section 5.1 in the appendix

$$-\mathcal{L}_{\omega_0}(u) = \log \mathbb{E}_{\psi_0}(e^{-(u(x_1)+\dots+u(x_n))}) \geq -\mathbb{E}_{\psi_0}(u(x_1) + \dots + u(x_n)),$$

using Jensen's inequality in the last step. Moreover, by formula 5.5  $\mathbb{E}_N(u(x_1) + \dots + u(x_n)) = \int_X u \beta_u$ . Since  $\beta_u = \omega_0^n / V$  in the homogenous case (compare the proof of Corollary 2), this means that  $-\mathcal{L}_{\omega_0}(u) \geq 0$ . Hence,  $u$  violates the inequality referred to above.

We also recall the following basic cocycle property of the functional  $\mathcal{F}_{\omega_0} := \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$ :

$$(2.14) \quad \mathcal{F}_{\omega_{u_2}}(u_1) + \mathcal{F}_{\omega_{u_3}}(u_2) = \mathcal{F}_{\omega_{u_3}}(u_1),$$

which is a direct consequence of the corresponding cocycle properties of  $\mathcal{E}_{\omega_0}$  and  $\mathcal{L}_{\omega_0}$ . These latter properties in turn are immediately obtained by integrating the corresponding differentials along line segments in  $\mathcal{H}_{\omega_0}$  (compare [70]).

**2.4. Analytic torsion.** Let  $F$  be a holomorphic vector bundle over the Kähler manifold  $(X, \omega_0)$ . Given an Hermitian metric  $h$  on  $F$  the pair  $(h, \omega_0)$  induces, in the standard way, Hermitian products on the space  $\Omega^{0,q}(X, F)$  of smooth  $(0, q)$ -forms with values in  $F$ . The corresponding *analytic torsion*, first introduced by Ray-Singer, is defined as

$$\tau_F(h, \omega_0) := - \sum_{q=0}^n (-1)^q q \log \det \Delta_{\bar{\partial}}^{(q)}$$

(in our sign convention), where  $\Delta_{\bar{\partial}}^{(q)} := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$  is the Dolbeault Laplacian acting on  $\Omega^{0,q}(X, F)$  and  $\log \det \Delta_{\bar{\partial}}^{(q)} := - \frac{\partial \zeta^{(q)}}{\partial s} \Big|_{s=0}$ , where  $\zeta^{(q)}(s) =$

$\sum_i (\lambda_i^{(q)})^{-s}$  is the meromorphic continuation to  $\mathbb{C}_s$  of the zeta function for the *strictly* postive eigenvalues  $\{\lambda_i^{(q)}\}$  of  $\Delta_{\bar{\partial}}^{(q)}$  (see [16, 42] and references therein). We will be concerned mainly with the case when  $n = 1$ , where

$$\tau_F(h, \omega_0) = \log \det \Delta_{\bar{\partial}}^{(1)} = \log \det \Delta_{\bar{\partial}}^{(0)}$$

(using that the non-zero eigenvalues of  $\Delta_{\bar{\partial}}^{(0)}$  and  $\Delta_{\bar{\partial}}^{(1)}$  coincide) and study the dependence of  $\tau_F(h, \omega_0)$  on  $h$ . The following proposition is a special case of the general anomaly formula of Bismut-Gillet-Soulé [16]:

**Proposition 17.** *Let  $F$  be a line bundle over a complex curve  $X$ . Let  $\omega_0$  be an Kähler metric on  $X$  with constant Gauss curvature (as a metric on  $TX$ ). Assume that  $L := F - K_X$  is ample and that  $\omega_0$  is normalized so that  $\omega_0 \in c_1(L)$ . Then*

$$\tau_F(e^{-u}h_0, \omega_0) - \tau_F(h_0, \omega_0) = \frac{\deg(K_X)}{2\deg(L)} J(u) + N\mathcal{F}_{\omega_0}(u),$$

where  $h_0$  is a metric on  $F$  whose normalized curvature form equals  $\omega_0$  and  $N = \dim H^0(X, F)$ .

*Proof.* It will convenient to use weight notation, i.e.  $h_0 = e^{-\psi_0}$  etc. The Hermitian product on  $H^0(X, L + K_X)$  induced by a given weight  $\psi = \psi_0 + u$  on  $L$  may be expressed as

$$\langle s, t \rangle_{\psi} := i \int s \wedge \bar{t} e^{-\psi} = \int s \bar{t} e^{-(\psi + \psi_{\omega_0})} \omega_0,$$

where  $\psi_{\omega_0} := \log \omega_0$  defines a weight on  $K_X$  such that its curvature  $dd^c \psi_{\omega_0} := -\text{Ric} \omega_0$ . In other words, the Hilbert space  $H^0(X, L + K_X)$  (with the Hermitian product determined by  $\psi$ ) is unitary equivalent to the Hilbert space  $H^0(X, F)$  determined by the weight  $\phi := \psi + \psi_{\omega_0}$  on  $F$  and the metric  $\omega_0$  on the tangent bundle  $TX$ . Next, we will use the anomaly formula of Bismut-Gillet-Soulé [16] for the Quillen metric on the determinant line  $\bigwedge^N H^0(X, F)$  (note that  $H^1(X, F)$  is trivial since  $L$  is ample). It reads as follows in our notation

(2.15)

$$\tau_F(e^{-u}h_0, \omega_0) - \tau_F(h_0, \omega_0) = \int_X \text{Td}(X, \omega_0) \wedge \tilde{c}h(\phi_0 + u, \phi_0) - N\mathcal{L}_{\omega_0}(u),$$

where

$$\text{Td}(X, \omega_0) = (1 + \frac{1}{2} \text{Ric} \omega_0) = (1 - \frac{\deg(K_X)}{2\deg(L)} \omega_0)$$

is the *Todd class* of  $TX$  represented by the constant curvature metric  $\omega_0 \in c_1(L)$  and

$$\tilde{c}h(\phi_1, \phi_0) = (\phi_1 - \phi_0) \left( 1 + \frac{dd^c \phi_1 + dd^c \phi_0}{2} \right)$$

is the *Bott-Chern class* of the pair of weights  $(\phi_1, \phi_0)$  on  $F$  associated to the *Chern character* of  $F$ . Since by assumption  $dd^c \psi_{\omega_0} = -\text{Ric} \omega_0 =$

$\frac{\deg(K_X)}{\deg(L)}\omega_0$  this means that

$$\tilde{c}h(\phi_0 + u, \phi_0) = u(1 + \frac{\omega_u + \omega_0(1 + \frac{\deg(K_X)}{\deg(L)})}{2}).$$

Expanding the integrand in 2.15 hence gives

$$\tau_F(e^{-u}h_0, \omega_0) - \tau_F(h_0, \omega_0) = \frac{\deg(K_X)}{2\deg(L)} \int_X u\omega_0 + \deg(L)\mathcal{E}_{\omega_0}(u) - N\mathcal{L}_{\omega_0}(u).$$

Now by the Riemann-Roch theorem  $N = \deg(F) - \deg(K_X)/2 = \deg(L) + \deg(K_X)/2$ , which finally proves the proposition using the relation 2.8.  $\square$

*Remark 18.* In the general anomaly formula in [16] the metric  $\omega_0$  is allowed to vary as well. In particular, when  $L = \mathcal{O}(0)$  is the trivial holomorphic line bundle over  $S^2$ , the metric  $h = 1$  is kept constant, but the conformal metric  $g_u = e^{-u}g_0$  on  $TS^2$  varies with  $u$ , the anomaly formula in [16] is equivalent to Polyakov's formula and then  $\log(\frac{\det \Delta_{g_u}}{\det \Delta_{g_0}})$  coincides with the functional  $\mathcal{F}_0$  (up to a multiplicative constant) [25].

Next, we follow standard notation in Kähler geometry and denote by  $h_{\omega_0}$  any Ricci potential of  $\omega_0$ , i.e.

$$(2.16) \quad \text{Ric } \omega_0 - \mu\omega_0 = dd^c h_0.$$

**Proposition 19.** *Fix  $\mu \in \{0, -1, 1\}$  and assume that  $L := \mu K_X$  is ample line bundle or trivial (if  $\mu = 0$ ). Then  $(\tau_F(e^{-ku}h_0, \omega_0) - \tau_F(h_0, \omega_0))/N_k = \mathcal{F}_{k\omega_0}(u) +$*

$$+ \frac{1}{2 \text{Vol}(L)} \left( \int h_{\omega_0} \left( \frac{\omega_0^n}{n!} - \frac{\omega_u^n}{n!} \right) - \mu(I_{\omega_0}(u) - J_{\omega_0}(u)) + O\left(\frac{1}{k}\right)J_{\omega_0}(u) + O\left(\frac{1}{k}\right) \right)$$

where the error term  $O(\frac{1}{k})$  means that  $|O(\frac{1}{k})| \leq C/k$ , where  $C$  only depends on  $\omega_0 \in c_1(L)$ .

*Proof.* By scaling invariance we may assume that  $\sup_X u = 0$ . Applying the anomaly formula as above gives the formula in the proposition but with a sum of error terms of the form  $O(\frac{1}{k})$  times

$$\int_X (-u(\omega_u)^i \wedge (\omega_0)^j \wedge \beta_{i,j}^{n-(i+j)})$$

where  $i + j < n$  and  $\beta_{i,j}$  is an  $(n - (i + j), n - (i + j))$  form depending on  $\omega_0$  only. Since there is an alternative way to get the explicit form of the leading term we skip the calculation (compare the remark below). Now, by a well-known inequality for weakly positive forms [44] there is a constant  $C$  such that

$$-C(\omega_0)^{n-(i+j)} \leq \beta_{i,j} \leq C(\omega_0)^{n-(i+j)}$$

Hence, we can conclude the proof of the proposition by observing that, since  $\sup_X u = 0$ ,

$$|\int_X (u(\omega_u)^i \wedge (\omega_0)^{n-i})| \leq C(J_{\omega_0}(u) - \int u \omega_0^n)$$

using formula 2.8. Finally, we just use the basic fact that  $u \mapsto \int u \omega_0^n$  is a bounded functional on the subspace of  $\mathcal{H}_{\omega_0}$  where  $\sup_X u = 0$  (see [43]).  $\square$

*Remark 20.* Letting  $k$  tend to infinity in the previous formula gives

$$\begin{aligned} & \lim_{k \rightarrow \infty} (\tau_F(e^{-ku} h_0, \omega_0) - \tau_F(h_0, \omega_0)) / k^n = \\ & = -\mathcal{S}(\frac{\omega_u^n}{\text{Vol}(L)n!}, \frac{\omega_0^n}{\text{Vol}(L)n!}) := -\frac{1}{2\text{Vol}(L)} \int_X \log(\frac{\omega_u^n}{\omega_0^n}) \frac{\omega_u^n}{n!} \end{aligned}$$

which gives a new proof of a result in [17] (which is valid for any ample line bundle). To see this one first uses that  $\mathcal{F}_{k\omega_0}(u)$  converges to  $-\mathcal{M}_{\omega_0}$  where  $\mathcal{M}_{\omega_0}$  denotes Mabuchi's  $K$ -energy with our normalizations (formula 4.3). Finally, by the explicit formula of Tian ([70], page 95) and Chen [26] for  $\mathcal{M}_{\omega_0}$  associated to an ample line bundle  $L$  this then proves the asymptotics above. It is interesting to see that the limiting functional (i.e. minus the relative entropy of the corresponding probability measures) is always bounded from above and maximized precisely for  $u$  such that  $\omega_u^n = \omega_0^n$  (by Jensen's inequality), which in turn means that  $u$  is constant by well-known uniqueness results for the Monge-Ampère equation.

### 3. PROOFS OF THE MAIN RESULTS

**3.1. Proof of Theorem 1.** By the cocycle property of  $\mathcal{F}_{\omega_0}$  (see [12, 11]) we may without loss of generality assume that  $u = 0$  is critical. Take a *continuous* element  $u_1$  in  $\mathcal{H}_{\omega_0}$  and the corresponding  $\mathcal{C}^0$ -geodesic  $u_t$  connecting  $u_0 = 0$  and  $u_1$ . Since  $u_t$  is a continuous path, combining Theorem 12 and Proposition 15 gives that  $\mathcal{F}_{\omega_0}(t) := \mathcal{F}_{\omega_0}(u_t)$  is a continuous concave function on  $[0, 1]$ . Hence, the inequality in Theorem 1 will follow once we have shown that

$$(3.1) \quad \frac{d}{dt}_{t=0+} \mathcal{F}(u_t) \leq 0.$$

Of course, if  $u_t$  were known to be a *smooth* path then this would be an immediate consequence of the assumption that  $u_0$  is critical combined with the chain rule (which would even yield equality above). To prove 3.1 first observe that by the concavity in Prop 15

$$(\mathcal{E}_{\omega_0}(u_t) - \mathcal{E}_{\omega_0}(u_0))/t \leq \frac{1}{t} \int_X (u_t - u_0)(\omega_{u_0})^n / n!$$

Hence, the monotone convergence theorem applied to the sequence  $(u_t - u_0)/t$  which decreases to the right derivative  $v_0$  of  $u_t$  at  $t = 0$  (using that

$u_t$  is convex in  $t$ ) gives

$$(3.2) \quad \frac{d}{dt}_{t=0+} \mathcal{E}_{\omega_0}(u_t) \leq \int_X v_0(\omega_{u_0})^n/n!$$

Hence,

$$\frac{d}{dt}_{t=0+} \mathcal{F}(u_t) \leq \int_X ((\omega_{u_0})^n/n! - \beta_{u_u})v_0 = 0,$$

where we have also used the dominated convergence theorem to differentiate  $\mathcal{L}_{\omega_0}(u_t)$  (compare [12, 11]). This finishes the proof of 3.1 and hence the first statement in the theorem follows.

*Uniqueness:*

*Step 1 ( $u_t$  is critical for all  $t$ ):* Assume now that  $u_1$  is a smooth maximizer of  $\mathcal{F}_{\omega_0}$  on  $\overline{\mathcal{H}}_{\omega_0}$  i.e. that  $\mathcal{F}_{\omega_0}(u_1) = \mathcal{F}_{\omega_0}(u_0)$  by the previous step. Since  $\mathcal{F}_{\omega_0}(t) := \mathcal{F}_{\omega_0}(u_t)$  is continuous and concave it follows that  $u_t$  maximizes  $\mathcal{F}_{\omega_0}$  on  $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}_{\mathbb{C}}^{1,1}(X)$  for all  $t$ . Next, we will show that  $u_t$  satisfies the Euler-Lagrange equation 1.6 for any fixed  $t$  (see [12] for similar arguments). To this end fix  $t = t_0$  and set  $u_{t_0} := u$ . Given a smooth function  $v$  on  $X$  consider the function  $f(t) := \mathcal{E}_{\omega_0}(P_{\omega_0}(u + tv)) - \mathcal{L}_{\omega_0}(u + tv)$  on  $\mathbb{R}_t$ . Since, the functional  $\mathcal{L}_{\omega_0}$  is increasing on  $\mathcal{C}^0(X)$  we have  $f(t) \leq \mathcal{F}_{\omega_0}(P_{\omega_0}(u + tv))$ . By assumption this means that the maximal value of the function  $f(t)$  is attained for  $t = 0$  (also using that  $P_{\omega_0}u = u$ ). In particular, since by Theorem 14  $f(t)$  is differentiable  $df/dt = 0$  at  $t = 0$  and Theorem 14 and formula 2.4 hence show that the Euler-Lagrange equation 1.6 holds (since it holds when tested on any smooth function  $v$ ).

*Step 2 ( $U \in \mathcal{C}^\infty(\dot{M})$ ):* (where  $\dot{M}$  denotes the interior of  $M$ ). By Theorem 10  $U$  is in  $\mathcal{C}_{\mathbb{C}}^{1,1}(M)$ . Moreover, by the homogenous Monge-Ampère equation 2.1 and the Euler-Lagrange equation 1.6 we have

$$(dd^c(U + |w|^2) + \pi_X^* \omega_0)^{n+1} = i\beta_u \wedge dw \wedge d\bar{w}$$

Hence, the following equation holds locally on  $\mathbb{C}^{n+1}$  (where we for simplicity have kept the notation  $U$  for the function obtained after subtracting a smooth and hence harmless function from  $U$ ) :

$$(3.3) \quad \det(\partial_{\zeta_i} \partial_{\bar{\zeta}_j} U) = e^{-U} \rho,$$

where  $\rho$  is a positive smooth function, depending on  $U$  (compare the discussion below formula 2.5). In particular,  $\det(\partial_{\zeta_i} \partial_{\bar{\zeta}_j} U)$  is locally in  $\mathcal{C}_{\mathbb{C}}^{1,1}$ . But then Theorem 2.5 in [18], which is a complex analog of a result of Trudinger for fully non-linear elliptic operators (compare Evans-Krylov theory), gives that  $U$  is locally in the Hölder space  $\mathcal{C}^{2,\alpha}$  for some  $\alpha > 0$ . Now the equation 3.3 shows that  $\det(\partial_{\zeta_i} \partial_{\bar{\zeta}_j} U)$  is also in  $\mathcal{C}^{2,\alpha}$ . Finally, since we have hence shown that  $U \in \mathcal{C}^2$ , standard theory of uniformly elliptic operators then allows us to boot strap using 3.3 and deduce that  $U \in \mathcal{C}^\infty$  locally (see Theorem 2.2 in [19]).

*Step 3: the inequalities*

$$(3.4) \quad 1/C' \omega_0 \leq \omega_{u_t} \leq C' \omega_0$$

*hold.* To see this first note that by the Euler-Lagrange equation 1.6 we have a uniform lower bound  $\omega_{u_t}^n > \delta \omega_0^n$  (also using the lower bound in formula 5.6 in the appendix). Combining the previous lower bound with the upper bound  $\omega_{u_t} \leq C\omega_0$  from Theorem 10 then shows that there is a positive constant  $C'$ , independent of  $t$ , such that 3.4 holds.

*Step 4: Application of “strict convexity”:*

By the above arguments  $\mathcal{F}_{\omega_0}(u_t)$  and  $\mathcal{E}_{\omega_0}(u_t)$  are both affine (and even constant) and hence it follows that  $\mathcal{L}_{\omega_0}(u_t)$  is affine. In case  $U$  were smooth *up to the boundary* of  $M$  applying Theorem 12 would hence prove the uniqueness statement in Theorem 1. To prove the general case we may without loss of generality assume that  $u_t(x)$  is smooth on  $[0, 1[ \times X$  (otherwise we just apply the same argument on  $[1/2, 1[$  and  $]0, 1/2]$ ). For any  $\epsilon > 0$  Theorem 12 (see also Theorem 2.6 in [15]) furnishes a 1-parameter holomorphic family  $S_t$  in  $\text{Aut}_0(X, L)$  with  $t \in [0, 1 - \epsilon]$  defined by the ordinary differential equation

$$(3.5) \quad \frac{dS_t(x(t))}{dt} = d_X(S(x(t))[V_t]_{x(t)})$$

with the initial data  $S_0 = I$  (the identity), where  $V_t$  is the vector field on  $X$  of type  $(1, 0)$  defined by the equation 2.6. As shown in [15]  $V_t$  defines a holomorphic vector field on the interior of  $A \times X$ . Furthermore, as shown in [15]

$$(3.6) \quad \psi_t - S_t^* \psi_0 = C_t$$

where  $\psi_t = \psi_0 + u_t$  and  $C_t$  is a constant for each  $t$ , i.e.

$$(3.7) \quad \omega_{u_t} = S_t^* \omega_0.$$

Now, by the bound 3.4 on  $\omega_{u_t}$  the point-wise norm of the vector field  $V_t$  wrt the metric  $\omega_0$  is uniformly bounded in  $t$  on all of  $X$ . Hence, the equation 3.5 and a basic normal families argument applied to the family  $S_t$  yields a subsequence  $S_{t_j}$  and a holomorphic map  $S_1$  on  $X$  such that  $S_{t_j}(x) \rightarrow S_1(x)$  uniformly on  $X$  (wrt the distance defined by the metric  $\omega_0$ ) where  $S_1$  is a biholomorphism according to the relation 3.7. Finally, letting  $t_j \rightarrow 1$  in the relation 3.6 and using that  $u_t$  is continuous on  $[0, 1] \times X$  finishes the proof of the uniqueness statement in the theorem.

**3.2. Proof of Corollary 2.** First observe that we may assume that  $H^0(X, L + K_X)$  has a non-zero element (otherwise the corollary is trivially true). But since  $(X, L)$  is homogenous it then follows immediately that  $L + K_X$  is globally generated. Hence, the conditions in Theorem 1 are satisfied.

Assume now that  $\omega_0$  is invariant under the holomorphic and transitive action of  $K$  on  $X$ . Then it follows that 0 is a critical point. Indeed, the volume form  $\omega_0^n/n!$  is invariant under the action of  $K$  on  $X$  and so is the Bergman measure  $\beta(0)$  (since it is defined in terms of the  $K$ -invariant weight  $\psi_0$ ). Since the action of  $K$  is transitive and both measures are normalized it follows that the function  $(\omega_0^n/n!)/\beta(0)$  on  $X$  is constant and hence equal to one. In other words, 0 is a critical point and by



Theorem 1 the inequality in the statement of Corollary 2 then holds. Finally, the last statement of the corollary is a direct consequence of the uniqueness part of Theorem 1.

**3.3. Proof of Corollary 3.** Let us first prove the first statement of the corollary. Since  $\mathcal{C}^\infty(X)$  is dense in  $W^{1,2}(X)$  we may assume that  $u$  is smooth. First observe that

$$(3.8) \quad \mathcal{F}_{\omega_0}(u) \leq \mathcal{F}_{\omega_0}(P_{\omega_0}u).$$

To see this note that, since, by definition,  $P_{\omega_0}u \leq u$  the fact that  $\mathcal{L}_{\omega_0}$  is increasing immediately implies  $\mathcal{L}_{\omega_0}(u) \geq \mathcal{L}_{\omega_0}(P_{\omega_0}u)$ . Next, observe that by the cocycle property of  $\mathcal{F}_{\omega_0}(u)$

$$\mathcal{E}_{\omega_0}(u) = \mathcal{E}_{\omega_0}(P_{\omega_0}u) + \int_X (u - P_{\omega_0}u)(\omega_u + \omega_{P_{\omega_0}u})/2$$

But, since, as is well-known the measure  $\omega_{P_{\omega_0}u}$  is supported on the open set  $\{u > P_{\omega_0}u\}$  (cf. Prop. 1.10 in [11] for a generalization) we have that the last term above is equal to

$$\begin{aligned} \int_X (u - P_{\omega_0}u)(\omega_u - \omega_{P_{\omega_0}u})/2 &= \int_X (u - P_{\omega_0}u)(dd^c(u - P_{\omega_0}u)) = \\ &= - \int_X d(u - P_{\omega_0}u) \wedge d^c(u - P_{\omega_0}u) \leq 0, \end{aligned}$$

where we have integrated by parts in the last equality, which is justified since, for example, by Theorem [8]  $P_{\omega_0}u$  is in  $\mathcal{C}^{1,1}(X)$  (but using that  $P_{\omega_0}u$  is in  $\mathcal{C}^0(X)$  is certainly enough by classical potential theory). Hence,  $\mathcal{E}_{\omega_0}(u) \leq \mathcal{E}_{\omega_0}(P_{\omega_0}u)$  which finishes the proof of 3.8. Since,  $\omega_{P_{\omega_0}u} \geq 0$  uniform approximation let's us apply Corollary 2 to deduce

$$\mathcal{F}_{\omega_0}(u) \leq \mathcal{F}_{\omega_0}(P_{\omega_0}u) \leq 0$$

which proves the first statement of the corollary.

Finally, the uniqueness will follow from Corollary 2 once we know that a maximizer  $u$  of  $\mathcal{F}_{\omega_0}$  on  $W^{1,2}(S^2)$  is smooth with  $\omega_u > 0$ . By the previous step we may assume that  $\omega_u \geq 0$ . But since  $W^{1,2}(S^2)$  is a linear space containing  $\mathcal{C}^\infty(X)$  the Euler-Lagrange equations  $\omega_{u_0} + dd^c u = \beta(u)$  hold for the maximizer  $u$ . Since  $\beta(u) = e^{-u}\rho > 0$  with  $\rho$  smooth, local elliptic estimates for the Laplacian then show that  $u$  is in fact smooth with  $\omega_t > 0$ . All in all we have proved that

$$(3.9) \quad -\mathcal{L}_{k\omega_0}(u) \leq -\mathcal{E}_{k\omega_0}(u)$$

for  $L = k\mathcal{O}(1)$  with conditions for equality.

**3.3.1. Explicit expression.** To make the previous inequality more explicit note that, by definition,

$$\mathcal{E}_{k\omega_0}(u) := \frac{1}{2 \int k\omega_0} \int (udd^c u + u2k\omega_0) = \frac{1}{2k} \int udd^c u + \int u\omega_0$$

Moreover, since for  $X = \mathbb{P}^1$  we have  $K_X = -\mathcal{O}(2)$  it follows that  $L + K_X = \mathcal{O}(k-2) =: \mathcal{O}(m)$ . Under this identification the scalar product on  $H^0(X, L + K_X)$  may be written as

$$\langle s, t \rangle_{k\psi_0+u} = c \int s \bar{t} e^{-(u+m\psi_0)} \omega_0$$

using that  $\omega_0$  is a Kähler-Einstein metric, i.e.  $\omega_0(z) := dd^c \psi_0 = c e^{-2\psi_0} idz \wedge d\bar{z}$  for some numerical constant  $c$ . Since the functional  $\mathcal{L}$  is invariant under overall scaling in definition of the scalar product  $\langle \cdot, \cdot \rangle_\psi$  we may as well assume that  $c = 1$ . Hence, since  $N_m = m + 1$ , we have

$$(3.10) \quad \mathcal{L}_m(u) := (m+1) \mathcal{L}_{\omega_0, k}(u) = -\log \det(c_i c_j \int_{\mathbb{C}} \frac{z^i \bar{z}^j}{(1+z\bar{z})^m} e^{-u} \omega_0),$$

where  $c_i = (\int \frac{|z^i|^2}{(1+z\bar{z})^m} \omega_0)^{-1/2}$ . Hence, the inequality 3.9 may be expressed as

$$(3.11) \quad \log \det(c_i c_j \int_{\mathbb{C}} \frac{z^i \bar{z}^j}{(1+z\bar{z})^m} e^{-u} \omega_0) \leq -\frac{m+1}{(m+2)} \frac{1}{2} \int (udd^c u) - (m+1)u\omega_0.$$

In particular, when  $m = 0$  the inequality above reads

$$\log(\int_{S^2} e^{-u} \omega_0) \leq \frac{1}{4} \int (udd^c u) + \int u \omega_0.$$

Finally, to compare with the notation of Onofri [56], note that, by definition,  $dd^c u = \frac{i}{2\pi} \partial \bar{\partial} u$  and hence, integration by parts gives,

$$-\int udd^c u = \frac{1}{\pi} \frac{i}{2} \int \partial u \wedge \bar{\partial} u.$$

Moreover, in terms of a given local holomorphic coordinate  $z = x + iy$ , we have  $\frac{i}{2} \partial u \wedge \bar{\partial} u = \frac{1}{4} |\nabla u|^2 dx \wedge dy$ , where  $\nabla = (\partial_x, \partial_y)$  is the gradient wrt the local Euclidian metric. By conformal invariance we hence obtain  $-\int udd^c u = \frac{1}{4\pi} \int |\nabla u|^2 d\text{Vol}_g$  for *any* Riemannian metric  $g$  on  $S^2$  conformally equivalent to  $g_0$ . In particular, taking  $g$  as the usual round metric on  $S^2$  induced by its embedding as the unit-sphere in Euclidian  $\mathbb{R}^3$  finally gives

$$\log(\int_{S^2} e^{-u} d\text{Vol}_g / 4\pi) \leq \frac{1}{4} \int |\nabla u|^2 - u) d\text{Vol}_g / 4\pi,$$

using that  $\omega_0 = d\text{Vol}_g / 4\pi$ . This is precisely the inequality proved by Onofri [56].

#### 3.4. Proof of Theorem 4.

*Uniqueness.* First note that, by the basic short exact sequence for restriction to a divisor (here the point  $p$ ) [28],  $K_X + L$  is not globally generated iff there is a point  $p \in X$  such that  $H^1(L - L_p + K_X) \neq \{0\}$ , where  $L_p$  is the holomorphic line bundle which has a holomorphic section  $s_p$  vanishing to order one precisely at  $p$ . By Serre duality means that  $H^0(L_p - L) \neq \{0\}$ . But since  $\deg L \geq 1$  this happens precisely when

$\deg L = 1$  and  $L = L_p$ . Moreover, it then follows from the Riemann-Roch theorem that  $\dim H^0(X, L + K_X) =$

$$= \deg L + \frac{1}{2} \deg K_X = 1 + \frac{2g-2}{2} = \dim H^0(X, K_X)$$

and hence that

$$(3.12) \quad H^0(X, L + K_X) = H^0(X, K_X) \otimes s_p.$$

As a consequence the Bergman measure can be factorized as (after replacing  $s_p$  with  $s_p / \|s_p\|_{\psi_0+u}$ )

$$\beta_u = i|s_p|^2 e^{-(\psi_0+u)} \omega_0 f_u,$$

where

$$f_u = \frac{1}{g} \sum_i \alpha_i \wedge \bar{\alpha}_i / \omega_0 > 0$$

and where  $\alpha_i$  is a base of holomorphic  $(1, 0)$ -forms in  $H^{1,0}(X) = H^0(K_X)$  which is orthonormal with respect to the Hermitian product on  $K_X$  obtained by twisting the canonical Hermitian product by  $|s_p|_{\psi_0}^2 e^{-u}$  i.e.

$$\int_X \alpha_i \wedge \bar{\alpha}_j |s_p|_{\psi_0}^2 e^{-u} = \delta_{ij}$$

In particular, when  $(X, \omega_0)$  is a torus with its invariant metric we have  $f_u = 1$ . The equation for a critical point, normalized so that  $\mathcal{L}_{\omega_0}(u) = 0$  hence reads

$$(3.13) \quad \omega_0 + dd^c u = |s|_{\psi_0}^2 e^{-u} f_u \omega_0$$

Now assume that  $u_0$  and  $u_1$  both satisfy the previous equation (by basic linear elliptic theory we may assume that  $u_i$  are smooth). As will be shown in section 3.8 the geodesic  $U := u_t$  connecting  $u_0$  and  $u_1$  has suitable regularity properties. In particular  $u_t$  is smooth on  $X$ . Let  $V_t$  be the vector field of type  $(1, 0)$  defined by the relation 2.6 on  $X - \{p\}$ , where we recall that  $p = \{s_p = 0\}$ . It will be enough to prove that  $V_t = 0$  for all  $t$ . Indeed, it then follows immediately from the definition of  $V_t$  that  $\bar{\partial}(\partial_t u) = 0$  on  $X$  and hence  $\partial_t u$  is constant on  $X$  for all  $t$ . Integrating over  $t \in [0, 1]$  then show that  $u_1 - u_0$  is constant on  $X$ , proving the uniqueness.

**3.4.1. Proof of  $V_t = 0$ .** Just as in section 3.1 we have that  $\mathcal{L}(u_t)$  is affine wrt  $t$ . Hence Corollary 40 in the appendix yields a holomorphic section  $h^{1,0}$  of  $L + K_X$  such that  $h^{1,0} \wedge \bar{\partial}(\partial_t \psi_t) / \partial_X \bar{\partial}_X \psi_t$  defines a holomorphic section of  $L$  over  $X$ . In particular, this means that  $h^{1,0}(V_t)$  defines a holomorphic section of  $L$  over  $X - \{p\}$  and hence  $V_t$  is holomorphic on  $X - \{p\} - \{h^{1,0} = 0\}$ . But since, by definition,  $V_t$  is locally bounded on  $X - \{p\}$  it then follows, by Riemann's extension theorem, that it is holomorphic on all of  $X - \{p\}$ . Next, by the decomposition 3.12  $h^{1,0} = \gamma \otimes s_p$  close to  $p$ , where  $\gamma \in H^{1,0}(X)$  is non-vanishing close to  $p$ . Hence,  $V_t \otimes s_p$  defines an element of  $H^0(X, TX + L) = H^0(X, -K_X + L)$ . In the case when  $g \geq 2$  this latter space is trivial (since  $-K_X + L$  then has negative degree) and hence it follows that  $V_t = 0$  on  $X - \{p\}$  in this case.

Finally, in the case when  $g = 1$  we have that  $TX$  is trivial and hence  $H^0(X, TX + L) = H^0(X, L) = \mathbb{C}s$ , forcing  $V = C \frac{\partial}{\partial \bar{z}}$ , where  $\frac{\partial}{\partial \bar{z}}$  denotes an invariant holomorphic vector field on the torus  $X$  and  $C$  is constant on  $X$ . To see that  $C = 0$  we use Lemma 13 which says that  $V$  lifts to a holomorphic vector field  $\tilde{V}$  on  $L$  over  $X - \{p\}$ . The explicit formula for  $\tilde{V}$  in the proof of Lemma 13 now shows that the coefficients of  $\tilde{V}$  are locally bounded (since  $V$  and  $u$  are smooth on  $X$ ) and hence it follows from Riemann's extension theorem that  $\tilde{V}$  extends holomorphically over the fiber of  $L$  over  $p$  (which is of codimension one in the total space of  $L$ ). Hence,  $\tilde{V}$  is the generator of an automorphism in  $\text{Aut}_0(X, L)$ . But this latter group is trivial over the torus (as the corresponding automorphism in  $\text{Aut}_0(X)$  must leave the point  $p$ , determined by  $L$ , invariant). It follows that  $\tilde{V}$  is tangent to the fibers of  $L$  over  $X$ , i.e. its projection  $V$  to  $X$  vanishes. This hence finishes the proof of uniqueness. Alternatively, the vanishing of  $C$  above can also be seen directly from the defining equation for  $V$  which gives

$$\bar{\partial}(\partial_t u) = C |s|_{\psi_0}^2 d\bar{z}$$

on  $X - \{p\}$  and hence on all of  $X$  since  $\partial_t u \in \mathcal{C}^\infty(X)$ . But integrating over  $X$  and using Stokes theorem then forces  $C = 0$  and hence  $\partial_t u$  is constant on  $X$  proving uniqueness as before.

**3.4.2. Existence.** First note that the following *coercivity estimate* holds for some  $\epsilon > 0$  and some constant  $C > 0$  :

$$(3.14) \quad \mathcal{F}_{\omega_0}(u) \leq -\epsilon J(u), \quad J(u) := \frac{1}{2} \int du \wedge d^c u$$

for all smooth  $u$  on  $X$ . To see this it is convenient to use the formula 5.4 in the appendix. Estimating the density of the corresponding probability measure for  $\psi_0$  from above and using Fubini's theorem gives a constant  $A$  such that

$$-\mathcal{L}_\omega(u) = \log \mathbb{E}_{\psi_0}(e^{-(u(x_1)+\dots+u(x_n))}) \leq \log \int_X e^{-u} \omega + A$$

(a variant of this argument appears in [42]). Next, we may assume that  $u \int_X u \omega_0 = 0$ . By Fontana's generalization [40] of the Moser-Trudinger inequality on  $S^2$  we have that the left hand side above is bounded from above by  $J(u)/2 + B$  (i.e. just as in the case when  $X = S^2$ ). Hence,

$$-\mathcal{L}_\omega(u) \leq J(u) - \epsilon J(u) + A + B$$

for  $\epsilon = 1/2$ . Since, by assumption  $\text{Vol}(L) = 1$  this proves the coercivity estimate 3.14. Finally, the existence of a critical point now follows from basic variational arguments (just like in [72]).

*Remark 21.* For a general complex manifold  $X$  the coercivity estimate (in terms of Aubin's  $J$ -functional) on the subspace  $\mathcal{H}_{\omega_0}$  of  $\mathcal{C}^\infty(X)$  implies the existence of a solution  $u$  of the critical point equation of  $\mathcal{F}_\omega$  of finite energy; see remark below for the definition of finite energy ( $u$  is smooth when  $n = 1$ , by elliptic regularity). In fact, it is enough to assume

that  $\mathcal{F}_{\omega_0}$  is  $J_{\omega_0}$ -proper [12]. This existence result is proved by a slight modification of the proof in [12] concerning the case when  $L = -K_X$  (the coercivity then means that the Fano manifold  $X$  is analytically stable in the sense of Tian). The point is that the functional  $\mathcal{L}_{\omega_0}(u)$  is continuous on the subspace of all normalized  $\omega$ -psh functions such that  $J_{\omega_0}(u) \leq C$  (for the same reason as in the case  $L = -K_X$ ), which yields the existence of an extremizer  $u$  with finite energy. The fact that  $u$  satisfies the critical point equation then follows from the differentiability theorem 14. Essentially the same argument was used in the proof of the uniqueness in Theorem 1 (note that since we are optimizing  $\mathcal{F}_{\omega}$  over a convex set with boundary it is a priori far from clear that the extremizer satisfy the critical point equation).

Finally, a remark about finite energy:

*Remark 22.* Consider the setting of Theorem 1 and assume that there exists a (smooth) critical point, which we may assume is given by 0. Then the inequality furnished by the theorem, i.e.

$$\mathcal{F}_{\omega_0}(u) := \mathcal{E}_{\omega_0}(u) - \mathcal{L}_{\omega_0}(u) \leq 0$$

actually holds for all  $u$  in  $\mathcal{E}^1(X, \omega_0)$ , i.e. for all  $u$  in the convex set of all  $u$  in  $\overline{\mathcal{H}}_{\omega_0}$  with *finite energy*;  $\mathcal{E}(u) > -\infty$ , where

$$\mathcal{E}(u) := \inf_{u' \geq u} \mathcal{E}(u')$$

when  $u'$  ranges over all elements in  $\mathcal{H}_{\omega_0}$  such that  $u' \geq u$ . Equivalently,  $\int_X (\omega_u)^n = \text{Vol}(L)$  and  $-\int_X u (\omega_u)^n < \infty$  in terms of *non-pluripolar products* (see [12] and references therein). The inequality on all of  $\mathcal{E}^1(X, \omega_0)$  is simply obtained by writing  $u$  as a decreasing limit of elements in  $\mathcal{H}_{\omega_0}$  and using the continuity of  $\mathcal{E}$  and  $\mathcal{L}_{\omega_0}$  under such limits [12] (note that  $e^{-u}$  is integrable if  $\mathcal{E}(u) > -\infty$  [12]). Moreover, in the case when  $\text{Aut}_0(X, L)$  is discrete it can be shown that any maximizer of  $\mathcal{F}_{\omega_0}$  on  $\mathcal{E}^1(X, \omega_0)$ , is in fact equal to a constant. The proof is a simple adaptation of the argument in [12] concerning the case  $L = -K_X$ . It would be interesting to know if the general uniqueness statement in Theorem 1 also remains true in the larger class  $\mathcal{E}^1(X, \omega_0)$ ?

**3.5. Proofs of Corollary 5 and Corollary 6.** First we consider the first corollary where  $X$  has genus zero. We will use the notation from section 3.3.1, i.e.  $F = \mathcal{O}(m)$ ,  $K_X = \mathcal{O}(-2)$  and  $L = \mathcal{O}(m+2)$ . Applying the formula in Proposition 17 gives, since  $\text{Vol}(L) = m+2, \log(\frac{\det \Delta_u}{\det \Delta_0}) =$

$$= \frac{\deg(K_X)}{2\deg(L)} J_{\omega_0}(u) + N \mathcal{F}_{\omega_0}(u) \leq -\frac{1}{(m+2)} \int du \wedge d^c u + 0 \leq 0$$

In particular, the left hand side above vanishes precisely when the gradient of  $u$  does, i.e. when  $u$  is a constant. This hence finishes the proof of Corollary 5.

Finally, when  $X$  has genus one Proposition 17 says that  $\log(\frac{\det \Delta_u}{\det \Delta_0})$  is proportional to  $\mathcal{F}_{\omega_0}(u)$  and hence Corollary 6 follows from Theorem 4.

**3.6. Proof of Theorem 8.** *The equation (i) :* Consider the functional

$$\mathcal{F}_{\omega_0}^\delta(u) := \mathcal{F}_{\omega_0}(u) - \delta J(u) := (\mathcal{E}_{\omega_0}(u) + \log \int (e^{g_p - u}) \omega_0) - \delta J(u)$$

whose critical point equation is precisely equation 1.8 with  $\mu = 4\pi(1 + \delta)^{-1}$  under the normalization  $\int e^{g_p - u} = 1$ . Indeed, with our normalizations  $dd^c u / \omega_0 = \Delta u / 4\pi$ . As shown in the proof of the uniqueness in Theorem 4 (when  $L = L_p$ ) the first term  $\mathcal{F}_{\omega_0}(u)$  above is strictly concave along geodesics connecting critical points. Moreover, by the last point in Proposition 15  $J_{\omega_0}$  is convex along geodesics and hence  $\mathcal{F}_{\omega_0}^\delta(u)$  is strictly concave for  $\delta \geq 0$ , i.e.  $\mu \leq 4\pi$  proving uniqueness in this case just as before.

Finally, to prove uniqueness for  $\mu \leq 4\pi + \epsilon$  it is, by a standard application of the implicit function theorem, enough to prove that the linearization of equation 1.8 for  $\mu = 4\pi$  has zero as its unique solution. This follows immediately from Proposition 23 below (recall that in the present case  $n = 1$ ,  $V = 1$  and  $\beta_u = e^{g_p - u} / \int_X e^{g_p - u}$ ).

*The equation (ii) :* applying the implicit function theorem as above gives existence for  $\mu \in ]4\pi - \epsilon, 4\pi + \epsilon[$ . Next assume that  $\mu \in ]0, 4\pi[$ , i.e.  $0 < q < 1$  for  $q := \mu / 4\pi$ . It is equivalent to prove uniqueness of solutions to the equation

$$\omega_0 + dd^c u = (|s|_{\psi_0}^2 e^{-u})^q \omega_0$$

which describe the critical points of

$$\mathcal{F}_q(u) := \mathcal{E}_{\omega_0}(u) + \frac{1}{q} \log \int (|s|_{\psi_0}^2 e^{-u})^q \omega_0$$

which are normalized in the sense that  $\int (|s|_{\psi_0}^2 e^{-u})^q = 1$ , which will henceforth be assumed.

To this end we let (for  $q$  fixed)  $w := qu + (1 - q)g_p$  so that  $(|s|_{\psi_0}^2 e^{-u})^q = |s|_{\psi_0}^2 e^{-w}$ . Note that  $w \in \overline{\mathcal{H}_{\omega_0}}$ , i.e.  $\psi_0 + w$  is the weight of a singular metric on  $L$  whose curvature form  $\omega_w$  is positive in the sense of currents. By a standard regularization argument we may write  $g_p$  as a decreasing limit of  $g^{(\epsilon)} \in \mathcal{H}_{\omega_0}$  and let  $w^{(\epsilon)} = qu + (1 - q)g^{(\epsilon)}$ . Hence,

$$\log \int (|s|_{\psi_0}^2 e^{-u})^q \omega_0 = \mathcal{L}_{\omega_0}(w) = \lim_{\epsilon \rightarrow 0} \mathcal{L}_{\omega_0}(w_\epsilon)$$

and then it follows as before that  $\mathcal{F}_q(u_t)$  is concave along any  $\mathcal{C}^0$ -geodesic  $u_t$  (since  $\mathcal{E}_{\omega_0}(u_t)$  is affine and  $\log \int (|s|_{\psi_0}^2 e^{-u_t})^q \omega_0$  is a limit of concave functions). To prove uniqueness it will, as before, be enough to prove that  $sV_u$  defines an element of  $H^0(X, L)$  where  $V_{u_t}$  is the vector field associated to a geodesic of critical points  $u_t$  as before (since  $TX$  is trivial we will identify  $V_u$  with a function as before). Moreover, since we have assumed that  $q < 1$  it will be enough to prove the following

$$(3.15) \quad \text{Claim: } \bar{\partial} V_u = 0 \text{ on } X - \{p\}$$

Indeed,

$$|V_u s|_{\psi_0}^2 = \left| \frac{\bar{\partial}(\partial_t u)}{(|s|_{\psi_0}^2 e^{-u})^q} \right|^2 |s|_{\psi_0}^2 \leq C \frac{1}{(|s|_{\psi_0}^{2q-1})^2}$$

which is clearly in  $L^2(X - \{p\})$  when  $0 < q < 1$  and hence, if the claim above holds,  $V_u s$  extends over  $\{p\}$  to a global holomorphic section of  $L$ , by basic local properties of holomorphic functions. To prove the previous claim first observe that explicit differentiation gives (this is essentially a simple special case of the calculation appearing in the appendix taking advantage of the fact that  $H^0(X, L) = \mathbb{C}s$ ) :

$$\partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(w_t) = \int \beta(w_t) ((\partial_t \partial_{\bar{t}} q u_t) - \int |s \partial_t q u_t - C_t|_{\psi_0}^2 e^{-w_t} \omega_0)$$

where  $\beta(w_t) = (|s|_{\psi_0}^2 e^{-u})^q$  and the constant  $C_t$  is uniquely determined by requiring that

$$\langle s(\partial_t q u_t - C_t), s \rangle_{\psi_0 + w_t} = 0$$

i.e. that

$$\alpha_t := s(q \partial_t u_t - C_t)$$

be orthogonal to  $H^0(X, L)$ . In other words,  $\alpha_t$  is the  $L^2(e^{-(\psi_0 + w_t)})$  minimal solution of the  $\bar{\partial}$ -equation 5.9, with  $s = s_i$ , in the appendix. Now by a standard regularization argument (see for example below)

$$(3.16) \quad \int_X |\alpha|_{\psi_0}^2 e^{-w} \omega_0 \leq \int_X |\bar{\partial}(\partial_t u_t)|_{\omega_{qu}}^2 |s|_{\psi_0}^2 e^{-w} \omega_0 < \infty,$$

(note that  $\omega_{qu}$  is the part of the current  $\omega_w$  which is absolute continuous wrt  $\omega_0$ ). All in all this means that

$$\partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(w_t)_{t=0} \geq \int \beta(w) c(qu) = 0$$

where  $c(u)$  is the non-negative function appearing in equation 2.3 (which in fact vanishes since  $u_t$  is a  $\mathcal{C}^0$ -geodesic). Moreover, equality holds above precisely when equality holds in the estimate 3.16. As in the previous case of smooth weights we will prove that the latter equality implies the claim above. We will proceed by regularization: we replace  $w$  with  $w^{(\epsilon)}$  and accordingly for all objects defined in terms of  $w^{(\epsilon)}$ . Then from the explicit definition of  $\alpha^{(\epsilon)}$  it follows directly that

$$\|\alpha^{(\epsilon)}\|_{w^{(\epsilon)}} \rightarrow \|\alpha\|_w, \quad \liminf_{\epsilon} \|\bar{\partial} \alpha^{(\epsilon)}\|_{w^{(\epsilon)}, \omega_{w^{(\epsilon)}}} \leq \|\bar{\partial} \alpha\|_{w, \omega_{qu}}$$

where a subscript of the form  $w, \omega$  indicates that we are taking the point-wise norm along the fibers of  $L$  and  $T^{0,1}X$  wrt the metrics  $e^{-(\psi_0 + w)}$  and  $\omega$ , respectively. But since

$$\|\alpha^{(\epsilon)}\|_{w^{(\epsilon)}} \leq \|\bar{\partial} \alpha^{(\epsilon)}\|_{w^{(\epsilon)}, \omega_{w^{(\epsilon)}}}, \quad \|\alpha\|_w = \|\bar{\partial} \alpha\|_{w, \omega_{qu}}$$

it then follows that

$$\|\alpha^{(\epsilon)}\|_{w^{(\epsilon)}} = (1 + o(1)) \|\bar{\partial} \alpha^{(\epsilon)}\|_{w^{(\epsilon)}, \omega_{w^{(\epsilon)}}},$$

where  $o(1)$  denotes a sequence tending to zero when  $\epsilon \rightarrow 0$ . Repeating the arguments in the end of the proof of Proposition 39 in the appendix it then follows readily that

$$\|\bar{\partial}(sV_{w^{(\epsilon)}})\|_{w^{(\epsilon)}, \omega_{w^{(\epsilon)}}} \rightarrow 0$$

and hence  $V_u$  is holomorphic on  $X - \{p\}$  (since we may assume that  $w^{(\epsilon)} \rightarrow w$  uniformly with all derivatives on compacts of  $X - \{p\}$  and  $V_u = V_w$  there). This proves the claim 3.15 and hence finishes the proof of the proposition.

The following proposition was used in the previous proof:

**Proposition 23.** *Assume that either the assumptions in Theorem 1 hold and that  $\text{Aut}_0(X)$  is trivial or that  $X$  is a complex curve of genus at least one. Then the null-space of the linearization  $A_0$  of the operator*

$$u \mapsto (d\mathcal{F}_{\omega_0})_u := \frac{1}{Vn!}\omega_u^n - \beta_u$$

*at the unique critical point  $u_0$  of  $\mathcal{F}_{\omega_0}$  consists of the constant functions.*

*Proof.* At least formally, (i.e. if  $\mathcal{H}_{\omega_0}$  were finite dimensional) this would follow from the strict convexity of  $\mathcal{F}_{\omega_0}$ , modulo constants, along smooth geodesics in  $\mathcal{H}_{\omega_0}$ . Next, we will provide a rigorous proof which avoids the use of geodesics. To this end fix  $v \in \mathcal{C}^\infty(X)$  and observe that

$$\int_X A_0[v]v = \frac{\partial^2 \mathcal{F}_{\omega_0}(u_t)}{\partial^2 t}, \quad u_t = u_0 + tv$$

at  $t = 0$ , only using that  $t = 0$  is a critical point of  $\mathcal{F}_{\omega_0}(u_t)$ . Assume that  $A_0v = 0$ . Recall that  $\mathcal{F}_{\omega_0} = \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$  and note that, by formula 2.13  $\frac{\partial^2 \mathcal{E}_{\omega_0}(u_t)}{\partial^2 t}$  coincides with the right hand side in formula 5.11 at  $t = 0$  since  $u_0$  satisfies the critical point equation. As a consequence the argument leading up to the inequality 5.11 shows that equality holds in Hörmanders's  $L^2$ -estimate. But then it follows from the previous arguments (for  $n = 1$  see section 3.4.1) that the corresponding vector field  $V_0$  associated to  $u_t$  at  $t = 0$  vanishes on  $X$ , which by definition is equivalent to  $v$  being constant. This finishes the proof of the uniqueness result.  $\square$

*Remark 24.* The argument using the implicit function theorem above also shows the following fact of independent interest: if  $\mathcal{L}$  is a relatively ample line bundle over a holomorphic submersion  $\mathcal{X} \rightarrow S$  such that the fibers have no holomorphic vector fields and the functional  $\mathcal{F}_{\omega_{s_0}}$  defined over a given fiber  $X := \mathcal{X}_{s_0}$  has a critical point, then there also exist critical points for  $s$  close to  $s_0$ . In the case when  $\mathcal{L}$  is the relative anti-canonical line bundle, i.e. the critical points are Kähler-Einstein metrics, this is well-known (a similar argument appears in Aubin's continuity method [4, 70] on Fano manifolds).



**3.7. Alternative proof of regularity and uniqueness.** In this section we will show how to prove the “uniqueness” in Theorem 1 only using the regularity of the geodesics furnished by Theorem 11 and the theory of fully non-linear elliptic operators in  $n$  complex dimensions (applied to the Monge-Ampère operator on  $X$  as in [18]). In particular, this latter theory amounts to the basic *linear* elliptic estimates for the Laplacian when  $n = 1$  which will allow us to handle geodesics connecting degenerate metrics in section 3.8.

We will denote by  $W^{r,p}(X)$  the Sobolev space of all distributions  $f$  on  $X$  such that  $f$  and the local derivatives of total order  $r$  are in  $W^{0,p}(X) := L^p(X)$  (equivalently, all local derivatives of total order  $\leq r$  are in  $L^p(X)$ ); see [3]. If  $f$  is function on  $M = [0, 1] \times X$  we will write  $f_t \in W^{r,p}(X)$  *uniformly wrt*  $t$  if the corresponding Sobolev norms on  $(X, \omega_0)$  of  $f_t$  are uniformly bounded in  $t$ . We will also use the following basic facts repeatedly:

- If  $f$  is a function on  $M$  such that  $f_t \in W^{r,p}(X)$  *uniformly wrt*  $t$ , then the distribution  $f$  is in  $W^{r,p}(M)$  and the corresponding Sobolev norms on  $M$  are bounded.
- Partial derivatives of distributions commute
- If  $f, g \in W^{1,p}(X)$  for any  $p > 1$ . Then  $fg \in W^{1,p}(X)$  for any  $p > 1$  and Leibniz product rule holds for the distributional derivatives.

Note that as in section 3.1 it will be enough to prove that the geodesic  $u_t$  is smooth wrt  $(t, x)$  in the *interiour* of  $M$ . However, the arguments below will even give uniform estimates on the local Sobolev norms up to the boundary of  $M$ .

Assume now that the boundy data  $u_0$  and  $u_1$ , defining the geodesic  $u_t$  are in  $\mathcal{C}^{1,1}(X)$ . Since  $u_t$  is convex in  $t$  the right derivative (or tangent vector)  $v_t(x) := \frac{d}{dt}_+ u_t$  exists for all  $(t, x)$ .

**Lemma 25.** The right tangent vector  $v_t$  of  $u_t$  at  $t$  is uniformly bounded on  $M$ .

*Proof.* First observe that by the convexity in  $t$

$$u_t - u_0 \leq t(u_1 - u_0) \leq C_1 t,$$

using that  $u_0$  and  $u_1$  are continuos and hence uniformly bouned on  $X$  in the last step. Hence,  $v_t \leq C$ . To get a lower bound first observe that there is a “psh extension”  $\tilde{u}_t$  which is uniformly Lipshitz. Indeed, just take  $\tilde{u}_t := (1-t)u_0 + tu_1 + Ae^t$  for  $A \gg 1$ . Using that  $0 \leq dd^c u_0, dd^c u_1 \leq C\omega_0$  it is straight-forward to check that  $dd^c \tilde{U} + \pi^* \omega_0 \geq 0$  on  $M$  for  $A$  sufficently large. Since  $U$  is defined by the upper envelope 2.2 it follows that  $\tilde{u}_t \leq u_t$  and hence

$$u_t - u_0 \geq \tilde{u}_t - \tilde{u}_0 \geq C_2 t.$$

giving  $v_0 \geq C_2$ . Finally, by convexity we get  $C_2 \leq v_0 \leq v_t \leq C_1$  which proves the lemma.  $\square$

**Proposition 26.** *Let  $u_0$  be a critical point of  $\mathcal{F}_{\omega_0}$  on  $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^{1,1}(X)$ ,  $u_1$  an arbitrary element in  $\overline{\mathcal{H}}_{\omega_0} \cap \mathcal{C}^{1,1}(X)$  and  $u_t$  the geodesic connecting  $u_0$  and  $u_1$ . If  $\mathcal{L}_{\omega_0}(u_t)$  is affine, then there is an automorphism  $S_1$  of  $(X, L)$ , homotopic to the identity, such that  $u_1 - u_0 = S_1^* \psi_0 - \psi_0$ .*

*Proof. Step 1:*  $u_t \in \mathcal{C}^\infty(X)$ . First note that by Theorem 11  $u_t \in \mathcal{C}_\mathbb{C}^{1,1}(X)$ . Moreover, as shown in the beginning of section 3.1 it follows under the assumptions above that, for any  $t$ , the function  $u_t$  satisfies the Euler-Lagrange equations 1.6 on  $X$ . Hence, just as in section 3.1 Blocki's complex version of the regularity result of Trudinger, now applied to local patches of  $\{t\} \times X$  immediately gives that  $u_t \in \mathcal{C}^\infty(X)$  (when  $n = 1$  this follows from basic linear elliptic theory).

*Step 2:*  $\Delta_X v_t \in L^\infty(X)$  uniformly wrt  $t$ . Differentiating the Euler-Lagrange equation wrt  $t$  from the right gives

$$(3.17) \quad n dd^c v_t \wedge (\omega_t)^{n-1} = \frac{d\beta_{u_t}(x)}{dt} =: R[v_t],$$

in the sense of currents. Of course, this would follow immediately from the chain rule if  $u_t$  were smooth in  $(t, x)$ . In the present case the right hand side is handled as in Lemma 38 in the appendix. As for the left hand side it is (a part from the trivial case  $n = 1$ ) handled precisely as in Lemma in the proof of formula 6.11 in [12]. Moreover, lemma 38 in the appendix implies the bound

$$(3.18) \quad \|R[v_t]/(\omega_0)^n\|_{L^\infty(X)} \leq C \|v_t\|_{L^\infty(X)}$$

To see this, just note that

$$R[v] \leq 2 \|v\|_{L^\infty(X)} \int_X |K(x, y)|^2 e^{-(\psi(x) + \psi(y))} = 2 \|v\|_{L^\infty(X)} \beta_u,$$

using the well-known “reproducing property” of the Bergman kernel (formula 5.3 in the appendix). By formula 5.6 in the appendix this proves the inequality 3.18.

Now, since  $\omega_t > \delta\omega_0$ , formula 3.17 gives that the distribution  $\Delta_{\omega_t} v_t$ , where  $\Delta_{\omega_t}$  is the Laplacian on  $X$  wrt the metric  $\omega_t := \omega_{u_t}$ , is in  $L^\infty(X)$  uniformly wrt  $t$  and

$$\|\Delta_{\omega_t} v_t\|_{L^\infty(X)} \leq C \|v_t\|_{L^\infty(X)} \leq C',$$

by lemma 25.

*Step 3:*  $\Delta_M u \in W^{1,p}(M)$  for any  $p \geq 1$ . First observe that by step 1

$$(3.19) \quad \partial_z(\partial_{z_i} \partial_{\bar{z}_j} u) \in L^\infty(X),$$

uniformly wrt  $t$ . Also note that

$$(3.20) \quad \partial_t(\partial_{z_i} \partial_{\bar{z}_j} u) \in L^\infty(X),$$

uniformly wrt  $t$ . Indeed,  $\partial_t(\partial_{z_i} \partial_{\bar{z}_j} u) = \partial_{z_i}(\partial_{\bar{z}_j} \partial_t u) = (\partial_{z_i} \partial_{\bar{z}_j}) v_t \in L^p(X)$ , uniformly wrt  $t$ , for any  $p > 1$ , by step 2 and local elliptic estimates for

$\Delta_X$ . Next, we will use that the following identity proved in lemma 27 below:

$$(3.21) \quad \partial_t \partial_{\bar{t}} u = |V_t|_{\omega_t}^2 = |\partial_{\bar{z}} v_t|_{\omega_t}^2,$$

where  $|V_t|_{\omega_t}^2$  denotes the point-wise norm of  $V_t$  wrt the metric  $\omega_t$  (where we have used that  $\omega_t > 0$ ). First we have

$$(3.22) \quad \partial_z(\partial_t \partial_{\bar{t}} u) = \partial_z |\partial_{\bar{z}} v_t|_{\omega_t}^2 \in L^p(X),$$

uniformly wrt  $t$ , for any  $p > 1$  using Step 1 and Step 2 combined with local elliptic estimates on  $X$  for  $\Delta_X$ . Next,

$$(3.23) \quad \partial_t(\partial_t \partial_{\bar{t}} u) \in L^p(X),$$

uniformly wrt  $t$ . Indeed,  $\partial_t(\partial_t \partial_{\bar{t}} u) = \partial_t |\partial_{\bar{z}} \partial_t u|_{\omega_t}^2$  and since locally  $\partial_t \omega_t = \partial_t(\partial_z \partial_{\bar{z}} u)$  3.23 follows from 3.20 and 3.22 combined with Leibniz product rule. All in all this proves Step 3.

Now by Step 3 and elliptic estimates for the Laplacian we have  $u \in W^{3,p}(M)$ . In particular,  $u$  is locally in  $\mathcal{C}^2(M)$ . As a consequence the proof of Theorem 2.6 in [15] immediately gives that  $V_t$  is a holomorphic vector field on  $X$  for any  $t$ . Finally, we will recall a slight variant of the argument in [15] which shows that  $\partial_{\bar{t}} V_t = 0$  for  $V_t$  seen as a distribution on the interior of  $M$ . To simplify the notation we assume that  $n = 1$ , but modulo the change to matrix notation the case  $n > 1$  is the same. First we write 2.6 in the form

$$(3.24) \quad \omega V = \partial_{\bar{z}} \partial_t u,$$

where we have identified  $V$  and  $\omega$  with elements in  $L^p(M)$  for  $p >> 1$ . By Leibniz rule

$$\partial_{\bar{t}}(V\omega) = (\partial_{\bar{t}} V)\omega + V(\partial_{\bar{t}} \omega)$$

Next, observe that

$$\partial_{\bar{t}} \omega = \partial_{\bar{t}}(\partial_z \partial_{\bar{z}} u) = \partial_{\bar{z}}(\partial_{\bar{t}} \partial_z u) = \partial_{\bar{z}}(\omega \bar{V}),$$

using 3.24 in the last step. Hence, since, as shown above,  $\partial_{\bar{z}} V = 0$ , the two previous equations together give

$$\partial_{\bar{t}}(V\omega) = (\partial_{\bar{t}} V)\omega + \partial_{\bar{z}}(V\omega \bar{V}) = \partial_{\bar{z}}(\partial_{\bar{t}} \partial_t u),$$

also using 3.24 in the last step and commuting  $\partial_{\bar{z}}$  and  $\partial_{\bar{t}}$ . Since,  $V\omega \bar{V} = |V|_{\omega}^2$  it follows by 3.21 that  $(\partial_{\bar{t}} V)\omega = 0$ . But since,  $\omega > 0$  and  $(\partial_{\bar{t}} V)$  is in  $L^p(M)$  for all  $p > 1$  this forces  $(\partial_{\bar{t}} V) = 0$  a.e. on  $M$ . In particular,  $(\partial_{\bar{t}} V) = 0$  as a distribution on  $M$ . Hence, it follows that the distribution  $V_t$  is in the null-space of the  $\bar{\partial}$ -operator on  $M$ . By local elliptic theory it follows that  $V_t$  is smooth and hence holomorphic in the interior of  $M$ . Finally, the automorphism  $S_1$  is obtained precisely as in the end of section 3.1.  $\square$

In the previous proof we used the following

**Lemma 27.** Under the assumptions in the previous proposition the following holds:  $\partial_t \partial_{\bar{t}} u \in L^\infty(X)$  uniformly in  $t$  and

$$\partial_t \partial_{\bar{t}} u = \left| \bar{\partial}_X \partial_t u \right|_{\omega_{u_t}}^2.$$

*Proof.* By assumption the Monge-Ampère measure  $(dd^c U + \pi_X^* \omega_0)^{n+1}$  vanishes on  $M$ . Moreover, by Step 1 in the proposition above  $\Delta_X u_t \in C^\infty(X)$  for any  $t$  with bounds on the Sobolev norms which are uniform wrt  $t$ . Combining this latter fact with lemma 25 gives that  $U$  is Lipschitz on  $M$ . Finally, as shown in Step 2 in the proof of proposition above  $\Delta_X \partial_t u_t \in L^\infty(X)$  uniformly wrt  $t$ . We will next show that these properties are enough to prove the lemma. As the statement is local we may as well consider the restriction of  $u := U$  to an open set biholomorphic to a domain in  $\mathbb{C}^{n+1} = \mathbb{C}_t \times \mathbb{C}_z^n$ . Denote by  $u^\epsilon$  the local smooth function obtain as the convolution of  $u$  with a fixed local compactly supported smooth family of approximations of the identity. Expanding gives

$$(3.25) \quad (dd^c U + \pi_X^* \omega_0)^{n+1} = (\partial_t \partial_{\bar{t}} u^\epsilon - |\partial_{\bar{z}} \partial_t u^\epsilon|_{\omega_{u^\epsilon}}^2)(\omega_{u^\epsilon})^n \wedge dt \wedge d\bar{t}.$$

Now since, by assumption,  $|\partial_{\bar{z}} \partial_t u^\epsilon|_{\omega_{u^\epsilon}}^2 \leq C$  the second term tends to  $|\partial_{\bar{z}} \partial_t u|_{\omega_u}^2(\omega_u)^n \wedge dt \wedge d\bar{t}$  weakly when  $\epsilon \rightarrow 0$ . Moreover, by assumption  $u^\epsilon \rightarrow u$  uniformly locally and since the Monge-Ampère operator is continuous, as a measure, under uniform limits of psh functions [29] it will now be enough to prove that

$$(3.26) \quad (\partial_t \partial_{\bar{t}} u^\epsilon)(\omega_{u^\epsilon})^n \wedge dt \wedge d\bar{t} \rightarrow (\partial_t \partial_{\bar{t}} u)(\omega_u)^n \wedge dt \wedge d\bar{t}$$

weakly, where the right hand side is well-defined since  $\partial_t \partial_{\bar{t}} u_t$  defines a positive measure on  $\mathbb{C}^{n+1}$  and  $(\omega_{u_t})^n / \omega_0^n$  is continuous on  $\mathbb{C}^{n+1}$ . To this end fix a test function  $f$  i.e. a smooth and compactly supported function on  $\mathbb{C}^{n+1}$ . Then, with  $\int$  denoting the integral over  $\mathbb{C}^{n+1}$ ,

$$\int f(\omega_{u^\epsilon})^n (\partial_t \partial_{\bar{t}} u^\epsilon) \wedge dt \wedge d\bar{t} =: \int g_\epsilon (\partial_t \partial_{\bar{t}} u^\epsilon) = - \int (\partial_t g_\epsilon) (\partial_{\bar{t}} u^\epsilon)$$

By assumption  $(\partial_t g_\epsilon)$  and  $(\partial_{\bar{t}} u^\epsilon)$  tend to  $(\partial_t u)$  and  $(\partial_{\bar{t}} u)$ , respectively in  $L^p(X)$  for any  $p > 1$ , uniformly wrt  $t$  (more precisely by the assumption on  $\Delta_X u_t$  and the fact that  $u$  is Lipschitz). Hence, by Hölder's inequality

$$\int g_\epsilon (\partial_t \partial_{\bar{t}} u^\epsilon) \rightarrow - \int (\partial_t g) (\partial_{\bar{t}} u).$$

Finally, since  $(\partial_t g) \in L^\infty(X)$  uniformly wrt  $t$  (by the assumption on  $\Delta_X u_t$ ) and since  $\partial_t \partial_{\bar{t}} u$  defines a positive measure, Leibniz rule combined with the dominated convergence theorem gives (by a simple argument using a regularization of  $g$ )

$$- \int (\partial_t g) (\partial_{\bar{t}} u) = \int g (\partial_t \partial_{\bar{t}} u)$$

This proves 3.26 and hence finishes the proof of the lemma.  $\square$

### 3.8. The one-dimensional case with degenerate boundary data.

In this section we will show that in the one-dimensional case the argument in the previous section goes through without assuming strict positivity of the curvature forms of the boundary data. This fact was used in the proof of Theorem 4.

**Proposition 28.** *Assume that  $\dim X=1$  and let  $u_0$  and  $u_1$  be two smooth critical points of the functional  $\mathcal{F}_{\omega_0}$ . Then the geodesic  $u_t$  connecting  $u_0$  and  $u_1$  satisfies the following regularity properties:*

- $u_t$  and in particular  $\omega_t$  are smooth on  $X$  for each fixed  $t$
- $\partial_X u_t$  is smooth on  $X$
- and  $(\partial_t \partial_{\bar{t}} u)(\frac{\omega_t}{\omega_0})$  is locally bounded in the interior of  $M$  and  $U(t, x) := u_t(x)$  is locally in  $\mathcal{C}_\mathbb{C}^{1,1}$  in the interior of  $\{(t, x) \in M : \omega_t > 0\}$

*Proof.* The key point is that Step 1 and Step 2 in the proof in the previous section are still valid when  $n = 1$  (even if  $L + K_X$  is not globally generated). Indeed, as shown in the beginning of section 3.4  $u_t$  is continuous and satisfies the following non-linear elliptic equation on  $X$

$$\Delta_X u_t + 1 = f e^{-u_t}$$

where  $f$  is a non-negative smooth function and  $\Delta_X$  denotes the (normalized) Laplacian on  $X$  wrt  $\omega_0$ . In particular,  $\Delta_X u_t \in L^p(M)$  for all  $p > 1$ . Repeating the argument and using the standard elliptic estimates for  $\Delta_X$  finally proves that  $u_t \in \mathcal{C}^\infty(X)$  (with uniform bounds wrt  $t$  on the derivatives). This proves the first point in the proposition.

As for Step 2, differentiating the equation above wrt  $t$  also shows that  $v_t \in \mathcal{C}^\infty(X)$  (uniformly wrt  $t$ ), the point being that the linearized equation does not involve  $\omega_t$  when  $n = 1$ . This proves the second point in the proposition.

Finally, to prove the last point we may apply a slight variant of Lemma 27 obtained by multiplying formula 3.21 with the function  $(\frac{\omega_t}{\omega_0})$  and using the previous points.  $\square$

*Remark 29.* When  $n = 1$  any critical point with finite Dirichlet energy is in fact smooth. Indeed, by the Moser-Trudinger inequality  $e^{-u_t}$  is in  $L^p(X)$  for any  $p$  and hence the previous boot strapping argument still applies.

## 4. THE LARGE TENSOR POWER LIMIT: ANALYTIC TORSIONS AND MABUCHI'S K-ENERGY ENERGY

In this section we will consider the asymptotic situation when the ample line bundle  $L$  is replaced by a multiple  $kL$  for a large positive integer  $k$ . Building on [15] Berndtsson we will relate the large  $k$  asymptotics of  $\mathcal{F}_{k\omega_0}$  to *Mabuchi's K-energy*. The work [15] was in turn inspired by the seminal work of Donaldson [35] where a functional closely related to  $\mathcal{F}_{k\omega_0}$  was introduced (see section 4.3 below). We will give a new simple proof of the lower bound on Mabuchi's K-energy (Theorem 30 below)

which only uses the  $C^0$ -regularity of the geodesic connecting two given smooth points in  $\mathcal{H}_{\omega_0}$ . See [25] for a proof which uses the  $\mathcal{C}_\mathbb{C}^{1,1}$ -regularity (Theorem [25]) in the case when the first Chern class of  $X$  is assumed non-positive. We will also consider the analytic torsion for large tensor powers over a Fano manifold and give the proof of Theorem 7. Finally, some conjectures will be proposed.

Fixing  $\omega_0 \in c_1(L)$  we will take  $k\omega_0$  as the reference Kähler metric in  $c_1(kL)$ . Throughout the section  $u$  will denote an element in  $\mathcal{H}_{\omega_0}$ . We will make use of the following essentially well-known asymptotics for the differential of the functional  $\mathcal{L}_{k\omega_0}$  :

$$(4.1) \quad (d\mathcal{L}_{k\omega_0})_{ku} = \frac{k}{\text{Vol}(L)} \left( \frac{\omega_u^n}{n!} - \frac{1}{k} \left( \frac{1}{2} (\text{Ric}\omega_u - S\omega_u) \wedge \omega_u^{n-1} / (n-1)! + O(1/k) \right) \right) / n!$$

where  $S$  is certain invariant associated to  $(X, L)$ . Using formula 2.4 the proof of the previous formula is reduced to the well-known asymptotics of the Bergman measure on  $kL + E$ , where  $E$  is a given line bundle on  $E$ , due to Tian-Catlin-Zelditch. Here we take  $E = K_X$  (see [15]). In particular, we obtain

$$(4.2) \quad (d\mathcal{F}_k)_{ku} := \frac{1}{\text{Vol}(L)} \frac{1}{2} (\text{Ric}\omega_u - \omega_u) \wedge \frac{\omega_u^{n-1}}{(n-1)!} + o(1)$$

Following Mabuchi [54, 70] the  $K$ -energy (also called the *Mabuchi functional*) is defined, up to an additive constant, by letting  $-\mathcal{M}$  be the primitive on  $\mathcal{H}_{\omega_0}$  of the exact one-form defined as the leading term above, i.e. as  $\omega_u^n$  times the scalar curvature minus its average. Hence,  $u$  is a critical point of  $\mathcal{M}$  on  $\mathcal{H}_{\omega_0}$  iff the Kähler metric  $\omega_u$  has *constant scalar curvature*. We will denote by  $\mathcal{M}_{\omega_0}$  the K-energy normalized so that  $\mathcal{M}_{\omega_0}(0) = 0$ . Integrating along line segments in  $\mathcal{H}_{\omega_0}$  and using 4.2 immediately gives the asymptotics

$$(4.3) \quad \mathcal{F}_k(u) = -\mathcal{M}_{\omega_0}(u) + o(1).$$

For the most general version of the following theorem see [27].

**Theorem 30.** *Assume that the Kähler metric  $\omega_{u_0}$  has constant scalar curvature. Then  $u_0$  minimizes Mabuchi's  $K$ -energy  $\mathcal{M}_{\omega_0}$  on  $\mathcal{H}_{\omega_0}$ .*

*Proof.* By the cocycle property of  $\mathcal{M}_{\omega_0}$  we may as well assume that  $u = 0$  in the statement above. Now fix an arbitrary  $u$  in  $\mathcal{H}_{\omega_0}$  and take the  $C^0$ -geodesics  $u_t$  connecting 0 and  $u$ . Given a positive integer  $k$  the fact that  $\mathcal{F}_k$  is concave along  $u_t$  (compare the proof of Theorem 1) immediately gives

$$\mathcal{F}_k(u) \leq \mathcal{F}_k(0) + \frac{d}{dt}_{t=0+} \mathcal{F}_k(u_t)$$

Combining formulas 4.3, 4.2 then gives

$$(4.4) \quad \mathcal{F}_k(u) \leq \mathcal{F}_k(0) + \frac{1}{\text{Vol}(L)} \frac{1}{2} \int (\omega_0 - \text{Ric}\omega_0) \wedge \left( \frac{\omega_0^{n-1}}{(n-1)!} + O(k^{-1}) \right) \omega_0^n (-v_0),$$

where  $v_0 = \frac{du}{dt}|_{t=0+}$ . By the assumption on  $\omega_0$  the integral above vanishes. Moreover, by lemma 25 we have that  $v_0$  is uniformly bounded (in fact it is enough to know that its  $L^1$ -norm is uniformly bounded, which can be proved as in the proof of Theorem 7 below. Letting  $k$  tend to infinity the assumption on  $u$  hence gives, also using 4.3,

$$-\mathcal{M}_{\omega_0}(u) \leq -\mathcal{M}_{\omega_0}(0),$$

which hence finishes the proof of the theorem.  $\square$

In particular, the proof above shows that,  $\mathcal{M}_{\omega_0}$  is “convex along a geodesic”, in the sense that it is the point-wise limit of the *convex* functionals  $\mathcal{F}_k$  along a geodesic connecting two points in  $\mathcal{H}_{\omega_0}$ , only using the  $C^0$ -regularity of the corresponding geodesic. Note however that the definition of  $\mathcal{M}_{\omega_0}$  as given above does not even make sense unless  $u_t$  is in  $\mathcal{C}^4(X)$ , for  $t$  fixed and  $\omega_t > 0$  (the smoothness assumption may be relaxed to  $u_t \in \mathcal{C}_\mathbb{C}^{1,1}(X)$  using the alternative formula for  $\mathcal{M}_{\omega_0}$  from [69, 26]). In the case when the geodesic  $u_t$  is assumed *smooth* and  $\omega_t > 0$  the argument in the proof of the theorem above is essentially contained in [15]. In this latter case the convexity statement seems to first have appeared in [53] (see also [33]). In [35] the previous theorem was proved using the deep results in [34] and the “finite dimensional geodesics” in approximations of  $\mathcal{H}_{\omega_0}$  as briefly explained below.

**4.1. Proof of Theorem 7.** Set  $L = -K_X$  so that  $S = 1$  and let  $\mathcal{T}_k(u) = (\tau_F(e^{-ku}h_0, \omega_0))/N_k$  in terms of the notation in section 2.4.

*Step 1:* For any given  $\epsilon > 0$  there is a metric  $\omega_0 \in c_1(-K_X)$  such that

$$(4.5) \quad \mathcal{F}_k(u) \leq \frac{n}{2}(1 - R(X) + \epsilon)(J_{\omega_0}(u) + C_\epsilon)$$

for  $k \geq k_\epsilon$ .

To prove this we continue with the notation in the proof of Theorem 30 above. By scaling invariance we may assume that  $\sup u = 0$ . Now, by the convexity of  $u_t$  in  $t$  we have  $v_0 \leq u_1 - u_0 = u$  and hence  $-v_0 \geq 0$ . Take  $t > 0$  and  $\omega_0$  such that  $\text{Ric}\omega_0 > t\omega_0$ . Using 4.4 there is a constant  $C$  (depending on  $t$ ) such that  $\mathcal{F}_k(u) \leq$

$$\begin{aligned} &\leq \frac{n}{2}(1-t)\left(\frac{1}{\text{Vol}(-K_X)} + Ck^{-1}\right) \int \left(\frac{\omega_0^n}{n!}\right)(-v_0) = -\left(\frac{n}{2}(1-t) + Ck^{-1}\right) \frac{\partial \mathcal{E}_{\omega_0}(u_t)}{\partial t} \Big|_{t=0} = \\ &= -\left(\frac{n}{2\text{Vol}(-K_X)}(1-t) + Ck^{-1}\right) \mathcal{E}_{\omega_0}(u_1) \end{aligned}$$

using that  $\mathcal{E}_{\omega_0}(u_t)$  is affine and  $\mathcal{E}_{\omega_0}(u_0) = 0$  in the last step. Since we have assumed that  $\sup u = 0$  this means, by a standard argument, that there is a constant  $C$  such that 4.5 holds for *any*  $u$ .

*Step 2:* For any  $\omega_0 \in c_1(-K_X)$  there is a positive constant  $C$  such that

$$\mathcal{T}_k(u) \leq \left(-\frac{1}{2n} + Ck^{-1}\right)(J_{\omega_0}(u) + C) + \mathcal{F}_k(u)$$

This follows immediately from Prop 19 combined with the basic inequality 2.11.

Finally, combining the previous two steps shows that  $\mathcal{T}_k(u)$  is bounded from above for  $k$  sufficiently large as long as  $\frac{n}{2}(1 - R(X)) - \frac{1}{2n} < 0$ , i.e. if

$$R(X) > 1 - n^{-2}.$$

This proves the boundedness with the reference metric  $\omega_0$  depending on  $t$ . The case of a general fixed reference weight then follows immediately from the anomaly formula in [16] (this observation was already used in [42]).

In the case when  $X$  is a Fano surface (i.e.  $n = 2$ ) it follows from [68] that  $R(X) = 1$  unless  $X$  is equal to  $\mathbb{P}^2$  blown up in one or two points. In the first case  $R(X) = 21/28$  and in the second case  $R(X) = 21/25$  [52] and hence the condition on  $R(X)$  above is always satisfied when  $n = 2$ .

*Maximizers on  $\mathbb{P}^2$*  : First observe that in the case of  $\mathbb{P}^2$  the error term  $O(1/k)$  and  $h_{\omega_0}$  in the formula in Prop 19 vanish :

$$(4.6) \quad (\tau_F(e^{-ku}h_0, \omega_0) - \tau_F(h_0, \omega_0))/N_k = -\frac{1}{2\text{Vol}(-K_X)} (I_{\omega_0}(u) - J_{\omega_0}(u)) + \mathcal{F}_{k\omega_0}(u),$$

Indeed, since  $n = 2$  the error term is of the form  $\int u\beta^{2,2} + C$  for  $\beta$  a form of maximal degree on  $X$  depending only on  $\omega_0$ . Now, since  $\omega_0$  is invariant under the  $SU(n+1)$  action so is  $\beta^{2,2}$ , i.e. it is equal to a constant  $a$  times  $(\omega_0)^n$ . But since the analytic torsion is invariant under  $u \mapsto u + c$  it follows that  $a = 0$ . Finally, by the inequality 2.11 the term  $I_{\omega_0}(u) - J_{\omega_0}(u)$  is non-negative and vanishes if and only if  $J_{\omega_0}$  does, i.e. if and only if  $u$  is constant. Since, by Corollary 2  $\mathcal{F}_{k\omega_0}(u) \leq 0$  this hence finishes the proof.

It is also interesting to note that the following concavity property holds:

**Proposition 31.** *Let  $(X, \omega_0)$  be equal to  $\mathbb{P}^1$  or  $\mathbb{P}^2$  equipped with the Fubini-Study metric. Then the functional on  $\mathcal{H}_{k\omega_0}$  defined by the analytic torsion associated to  $-kK_X$  (with respect  $\omega_0$ ) is concave along smooth geodesics in  $\mathcal{H}_{k\omega_0}$  for any  $k \geq 0$ . Moreover, if  $(X, \omega_0)$  is a Kähler-Einstein manifold of semi-positive Ricci curvature of dimension  $n \leq 2$  then the large  $k$  limit of the analytic torsion functionals, i.e. minus the relative entropy  $u \mapsto \mathcal{S}(\mu_u, \mu_0)$  (see remark 20) is also concave along smooth geodesics.*

*Proof.* By assumption  $h_{\omega_0} = 0$ . By formula 4.6 (and its analogue in section 2.4 for  $n = 1$ ) combined with the concavity of  $\mathcal{F}_{k\omega_0}$  established in previous sections it is enough to observe that the functional  $I_{\omega_0} - J_{\omega_0}$  is convex along smooth geodesics when  $n \leq 2$ . To this end first note that when  $n = 1$  we have  $I_{\omega_0} - J_{\omega_0} = J_{\omega_0}/2$  which is convex according to Prop 15. As for the case  $n = 2$  a direct computation, using integration by parts and the geodesic equation gives

$$d^2(I_{\omega_0}(u_t) - J_{\omega_0}(u_t))/d^2t = \frac{1}{2\pi} \int \omega_0 \wedge (\partial_t \partial_t u_t \omega_{u_t} - i \partial_X(\partial_t u_t) \wedge \bar{\partial}_X(\partial_t u_t)).$$



Next, observe that

$$i\partial_X(\partial_t u_t) \wedge \bar{\partial}_X(\partial_t u_t) \leq |\bar{\partial}_X \partial_t u|_{\omega_{u_t}}^2 \omega_{u_t}$$

using the basic fact that a positive  $(1, 1)$  form  $\eta$  may be point-wise estimated from above by  $\text{tr}(\eta)\omega$  where  $\text{tr}(\eta)$  is the trace of  $\eta$  wrt the Kähler form  $\omega$ . Combining this inequality with the previous equation and the geodesic equation 2.3 for  $u_t$  finally proves the claimed convexity of  $I_{\omega_0}(u_t) - J_{\omega_0}(u_t)$ .  $\square$

*Remark 32.* The analytic torsion functionals are not bounded from above on the space of *all* smooth metrics on  $-K_X$ . Indeed, as explained in Remark 16 there are smooth functions  $u$  such that the functionals  $\mathcal{F}_k(tu)$  and  $-(I_{\omega_0}(tu) - J_{\omega_0}(tu))$  tend to infinity when  $t \rightarrow \infty$ . Hence, it follows from formula 4.6 that  $\mathcal{F}_k(u)$  also tends to infinity.

**4.2. Proof of a conjecture of Aubin.** Given a polarized manifold  $(X, L)$  we define an invariant  $B(X, L)$  as the following infimum over all Kähler metrics  $\omega \in c_1(L)$  :

$$B(X, L) := \inf_{\omega} B(X, \omega), \quad B(X, \omega_u) := \sup_X \left( \frac{\beta_u}{\omega_u^n / Vn!} \right).$$

Note that  $B(X) \geq 1$  since  $\beta_u$  and  $\omega_u^n / Vn!$  are both probability measures . When  $X$  is a Fano manifold we will write  $B(X) = B(X, -K_X)$ , i.e.

$$(4.7) \quad B(X) = \inf_{\omega} \sup_X (\exp(-h_{\omega}(x)))$$

where  $h_{\omega}$  is the “canonical” Ricci potential of  $\omega$  (i.e.  $h_{\omega}$  satisfies equation 2.16 with  $\mu = 1$  and  $\int_X e^{h_{\omega}} \frac{\omega_0^n}{Vn!} = 1$ ). Repeating the argument in the proof of step one in section 4.1, but now using that

$$\frac{d}{dt}_{t=0+} \mathcal{F}_{\omega_0}(u_t) = \int_X \left( \frac{\beta_u}{\omega_u^n / Vn!} - 1 \right) \omega_u^n / Vn! (-v_0)$$

immediately gives the following result of independent interest.

**Theorem 33.** *For any ample line bundle  $L \rightarrow X$  the following inequality holds (with the same notation as in Theorem 1): for any  $\epsilon > 0$  there is a constant  $C_{\epsilon}$  (also depending on  $\omega_0$ ) such that*

$$\mathcal{F}_{\omega_0}(u) \leq \frac{1}{V} (B(X, L) - 1 + \epsilon) J_{\omega_0}(u) + C_{\epsilon}$$

*In particular, when  $X$  is Fano and  $L = -K_X$  this means that for any  $\epsilon > 0$  there is a constant  $C_{\epsilon}$  such that*

$$(4.8) \quad \log \int_X e^{-u} \frac{\omega_0^n}{Vn!} \leq \frac{1}{V} (B(X) + \epsilon) J_{\omega_0}(u) + C_{\epsilon}$$

*for all  $u \in \mathcal{H}_{\omega_0}$  such that  $\int_X u \omega_0^n = 0$*

The second inequality 4.8 above answers in the affirmative the following conjecture of Aubin [4] (in the case when  $\omega_0 \in c_1(-K_X)$ ): there exists positive constants  $A$  and  $B$  such that

$$(4.9) \quad \log \int_X e^{-u} \frac{\omega_0^n}{V n!} \leq \frac{A}{V} J_{\omega_0}(u) + B,$$

for all  $u \in \mathcal{H}_{\omega_0}$  such that  $\int_X u \omega_0^n = 0$  (regardless if  $X$  admits a Kähler-Einstein metric or not). Such inequalities were further studied by Ding [30] who proved that one may take  $A = n + 1$  if one restricts to all  $u$  such that  $\text{Ric } \omega_u > \epsilon \omega_u$  for some  $\epsilon > 0$ . He also observed that the general conjectured inequality 4.9 of Aubin implies the existence of a solution  $\omega$  to Aubin's equation

$$\text{Ric } \omega = t\omega + (1 - t)\omega_0$$

for any given  $t \in [0, 1/A^n[$  (see Remark 2 on p.468 in [30]). Hence, applying Ding's argument gives (since clearly  $\text{Ric } \omega > t\omega$ ) the following Corollary of the previous theorem

**Corollary 34.** *Let  $X$  be a Fano manifold of complex dimension  $n$ . Then the following relation between the two invariants  $R(X)$  and  $B(X)$ , defined by formulas 1.7 and 4.7, respectively, holds:*

$$R(X) \geq 1/B(X)^n$$

In view of the previous inequality it would be interesting to know whether there is a *universal upper bound* on  $B(X)$  only depending on the *dimension*  $n$  of the Fano manifold  $X$ ? As pointed out by Tian-Yau [67] and Ding [30] such a universal lower bound on  $R(X)$  is equivalent to a universal upper bound on  $\text{Vol}(-K_X)$ . This latter bound was a well-known conjecture in algebraic geometry at the time, subsequently proved in [48] using Mori's "bend-and-brake" method of producing rational curves.

One approach to proving a universal upper bound on  $B(X)$  could be to study  $\liminf_{t \rightarrow \infty} B(X, \omega_t)$  for  $\omega_t$  evolving according to the (normalized) Kähler-Ricci flow [71]. By the fundamental estimate of Perelman for  $h_{\omega_t}$ , the previous limit is always finite and equal to one if  $X$  admits a Kähler-Einstein metric [71].

Finally, it should be pointed out that a different relation between  $R(X)$  and the properness of Mabuchi's  $K$ -energy has been established by Székelyhidi [63].

**4.3. Comparison with Donaldson's setting and balanced metrics.** In the setting of Donaldson [35] the role of the space  $H^0(X, L + K_X)$  is played by the space  $H^0(X, L)$ . Any given function  $u$  in  $\mathcal{H}_{\omega_0}$  induces an Hermitian norm  $\text{Hilb}(u)$  on  $H^0(X, L)$  defined by

$$\text{Hilb}(u)[s]^2 := \int_X |s|_{\psi_0}^2 e^{-u} (\omega_u)^n / n!$$

Then the functional that we will refer to as  $\mathcal{L}_D(u)$ , which plays the role of  $\mathcal{L}_{\omega_0}(u)$  in Donaldson's setting, is defined as in formula 1.4, but using the

scalar product on  $H^0(X, L)$  corresponding to  $\text{Hilb}(u)$ . In other words,

$$(4.10) \quad \mathcal{L}_D(u) = \mathcal{L}_{\omega_0 + \text{Ric}\omega_0}(u - \log(\omega_u^n / \omega_0^n))$$

It turns out that  $\mathcal{L}_D(u)$  is *concave* along smooth geodesics (see Theorem 3.1 in [15] for a generalization of this fact). However, it does not appear to be concave along a general psh paths, which makes approximation more difficult in this setting. Moreover, Theorem 2 in [35] says that the critical points of  $\mathcal{E}_{\omega_0} - \mathcal{L}_D$  are in fact *minimizers*.<sup>1</sup> On the other hand using the relation 4.10 Theorem 1 corresponds to an *upper* bound on  $\mathcal{E}_{\omega_0} - \mathcal{L}_D$  in terms of an explicit local functional (as long as  $\omega_0 + \text{Ric}\omega_0 \geq 0$  if  $n > 1$ ).

A major technical advantage of Donaldson's setting is that the critical points (which are called *balanced* in [35]) of the functional  $\mathcal{E} - \mathcal{L}_D$  acting on all of  $\mathcal{C}^\infty(X)$  are automatically of the form

$$(4.11) \quad \psi = \log\left(\frac{1}{N} \sum_i |S_i|^2\right)$$

for some base  $(S_i)$  in  $H^0(X, L)$ . In particular,  $u$  is automatically in  $\mathcal{H}_{\omega_0}$  (assuming that  $L$  is very ample). This is then used to replace the space  $\mathcal{H}_{\omega_0}$  by the sequence of *finite dimensional* symmetric spaces  $GL(N, \mathbb{C})/U(N)$  corresponding to the set of metrics on  $L$  of the form 4.11 (called Bergman metrics). In particular, the new geodesics, defined wrt the Riemannian structure in the symmetric space  $GL(N, \mathbb{C})/U(N)$  are automatically smooth and the analysis in [35] is reduced to this finite dimensional situation.

Note also that in this setting there is a sign difference in the expansion 4.1, where  $\text{Ric } \omega_u$  is replaced by  $-\text{Ric } \omega_u$ . As a consequence, in Donaldson's case the functional corresponding to  $\mathcal{F}_k$  converges to  $\mathcal{M}_{\omega_0}$  (without the minus sign!), which hence becomes *convex* along smooth geodesics, which is consistent with the conclusion reached above, as it must.

**4.4. Conjectures.** In the light of the work of Donaldson referred to above it is natural to make the following

**Conjecture 35.** *Assume that  $\text{Aut}_0(X)$  is trivial  $c_1(L)$  contains a Kähler metric  $\omega_0$  of constant scalar curvature. Then the following holds:*

- *for  $k$  sufficiently large the functional  $\mathcal{F}_k$  admits a critical point  $u_k$  in  $\mathcal{H}_{\omega_0}$*
- *The convergence  $\omega_k \rightarrow \omega_0$  as  $k \rightarrow \infty$  holds in the  $\mathcal{C}^\infty$ -topology.*

Note that when  $L$  is a degree one line bundle over a complex curve of genus at least one, then we have already established existence for  $k = 1$ . Since, loosely speaking existence also holds for  $k = \infty$  (since the limiting functional is minus Mabuchi's  $K$ -energy whose critical points are constant curvature metrics) we make the following more precise

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<sup>1</sup>Comparing with the notation in [35],  $\mathcal{L}_D$ ,  $\mathcal{E}$  and  $u$  correspond to  $-\mathcal{L}$ ,  $-I$  and  $-\phi$ , respectively.

**Conjecture 36.** *Let  $L \rightarrow X$  be an ample line bundle over a complex curve of genus at least one. Then the functionals  $\mathcal{F}_k$  admit a critical point for any  $k$ .*

In particular, the case when  $X$  has genus one would together with the results in the present paper confirm the conjecture of Gillet-Soulé, since  $\mathcal{F}_k$  coincides with the corresponding analytic torsion functional in this case.

Motivated by the case when  $L = -K_X$  [59] we further make the following

**Conjecture 37.** *Assume that  $\text{Aut}_0(X)$  is trivial and that  $\mathcal{F}_{\omega_0}$  admits a critical point in  $\mathcal{H}_{\omega_0}$ . Then  $\mathcal{F}_{\omega_0}$  is coercive (see remark 21).*

Note that if  $\mathcal{H}_{\omega_0}$  were finite dimensional then this would follow immediately from the strict concavity furnished by Theorem 12 (in this case the equivalence is closely related to the Kempf-Ness principle for stability in geometric invariant theory [66]).

**4.5. Relation to random sections and gauge theory.** The critical point equation and hence the conjectures above can be given a natural interpretation in terms of random sections. Indeed, as shown by Shiffman-Zelditch [61] the Bergman measure  $\beta_\phi$  of a weight  $\phi$  on  $L$  admits the following probabilistic interpretation (which is different from the one discussed in the appendix): let  $d\nu_\phi$  be the probability measure on  $H^0(X, L + K_X)$  defined as the Gaussian probability measure on the Hilbert space  $(H^0(X, L + K_X), \langle \cdot, \cdot \rangle_\phi)$ . Then

$$\beta_\phi = \mathbb{E}(s \wedge \bar{s} e^{-\phi}),$$

where  $\mathbb{E}$  denotes the expectation (i.e. average) over  $s \in H^0(X, L + K_X)$  wrt  $d\nu_\phi$  of the measure-valued random variable  $s \mapsto s \wedge \bar{s} e^{-\phi}$ . Then the critical point equation may be formulated as

$$(4.12) \quad (dd^c \phi)^n / n! V = \mathbb{E}(s \wedge \bar{s} e^{-\phi})$$

for a positively curved metric  $e^{-\phi}$  on  $L$ .

It may be illuminating to reformulate the previous equation in terms of gauge theory [37]. For concreteness we will only consider the case when  $X$  is a genus one curve, i.e. a one-dimensional torus and denote by  $\omega_0$  the standard invariant Kähler metric on  $X$ . Fix an Hermitian line bundle  $E \rightarrow X$  (i.e. with structure group  $U(1)$ ) of positive topological degree  $k$ . Consider the following equation for a unitary connection  $A$  on  $E$ :

$$(4.13) \quad \frac{i}{2\pi k} F_A = \mathbb{E}(|\Psi|^2) \omega_0$$

where  $F_A$  is the curvature two-form of  $A$  and where  $\mathbb{E}$  now denotes the expectation wrt the Gaussian probability measure on the  $k$ -dimensional Hilbert space

$$\mathcal{H}_A = \{ \Psi : \bar{\partial}_A \Psi = 0 \}$$

realized as a subspace of the space of all smooth sections  $\Psi$  with values in  $E$  equipped with the Hermitian product obtain by integrating against  $\omega_0$ . Here  $\bar{\partial}_A$  denotes the  $(0, 1)$ -part of the connection  $A$  (identified with a differential operator). A standard argument (see for example [37, 21]) then yields an isomorphism between the moduli space  $\text{UN}_k(X)$  of gauge equivalence classes of solutions  $A$  of equation 4.13 with the space of solutions  $\phi$  (modulo scaling) of the critical point equation for  $\phi$  when  $L$  ranges over all equivalence classes of degree  $k$  holomorphic line bundles over  $X$ . In this formulation the second point in conjecture 35 in this case is equivalent to the convergence of  $F_{A_k}/k$  towards  $\omega_0$  when  $A_k$  is in  $\text{UN}_k(X)$ .

Finally, it should be pointed out that when  $k = 1$  equation 4.13 is clearly equivalent to the following equation for  $(A, \Psi)$  :

$$(4.14) \quad \bar{\partial}_A \Psi = 0, \quad \frac{i}{2\pi} F_A = |\Psi|^2 \omega_0$$

In the physics litterature such  $(A, \Psi)$  describe the periodic static one-vortex solutions of the Jackiw-Pi model (also called non-topological non-relativistic self-dual *Chern-Simons-Higgs solitons* or the static *gauged non-linear Schrödinger equation*; see [1] and references therein.) In this formulation the uniqueness result in Corollary 8 (for  $\mu = 4\pi$ ) amounts to saying the solutions of 4.14 are uniquely determined, up to gauge equivalence, by the zero  $p$  of the corresponding “Higgs-field”  $\Psi$ . Note also that the equation corresponding to parameter  $\mu \in ]0, 4\pi]$  in Corollary 8 is obtained by replacing  $|\Psi|^2$  in equation 4.14 with  $t|\Psi|^2 + (1 - t)$  for  $\eta = 4\pi t$  for  $t > 0$ . For  $t$  *negative* this equation is equivalent to the *Yang-Mills-Higgs (Ginzburg-Landau) equations* on a Riemann surface [21] (after rescaling  $\Psi$ ). For a general Hermitian line bundle  $L$  of degree  $V > 0$  and an effective divisor  $D := \sum_i q_i p_i$  where  $p_i \in X$  and  $q_i$  is a positive integer the Yang-Mills-Higgs equations admit a unique gauge equivalence class of solutions  $(A, \Psi)$  such that  $D$  is the zero-divisor of  $\Psi$ . Note however that for the Chern-Simons-Higgs equation uniqueness fails when  $V = 2$  and  $D = 2p$  since the equation is then equivalent to equation 1.9 (compare the discussion below Theorem 8).

## 5. APPENDIX

**5.1. Bergman kernels, Toeplitz operators and determinantal point processes.** Given a function  $u$  corresponding to the weight  $\psi := \psi_0 + u$  on the line bundle  $L$  we denote by  $K_u(x, y)$  the *Bergman kernel* of the Hilbert space  $(H^0(X, L + K_X), \langle \cdot, \cdot \rangle_{\psi_0 + u})$ , i.e.

$$K_u(x, y) := i^{n^2} \sum_{i=1}^N s_i(y) \wedge \bar{s}_i(x),$$

represented in terms of a given orthonormal base  $(s_i)$  in  $(H^0(X, L + K_X), \langle \cdot, \cdot \rangle_{\psi_0 + u})$ . This kernel may be characterized as the integral kernel of

the corresponding orthogonal projection  $\Pi_u$  onto  $(H^0(X, L+K_X), \langle \cdot, \cdot \rangle_{\psi_0+u})$ , i.e. for any smooth section  $s$  of  $L + K_X$

$$(5.1) \quad (\Pi_u s)(x) = \int_{X_y} s(y) \wedge \bar{K}(x, y) e^{-\psi(y)}$$

The *Toeplitz operator*  $T[f]$  with symbol  $f \in C^0(X)$ , acting on  $(H^0(X, L+K_X), \langle \cdot, \cdot \rangle_{\psi_0})$  (defined below formula 1.5) may then be expressed as

$$(5.2) \quad (T[f])(x) = \int_{X_y} f(y) s(y) \wedge \bar{K}(x, y) e^{-\psi(y)}$$

Applying 5.1  $K_u(x, \cdot)$  gives the following “integrating out” formula

$$(5.3) \quad N\beta_u(x) := K_u(x, x) e^{-\psi(x)} := \int_{X_y} |K(x, y)|^2 e^{-(\psi(x)+\psi(y))}$$

When studying the dependence of  $\beta_u$  on  $u$  it is useful to express  $\beta_u(x)$  as the normalized *one-point correlation measure* of a determinantal random point process. To this end we recall that  $\psi = \psi_0 + u$  induces a probability measure  $\gamma$  on the  $N$ -fold product  $X^N$  defined by the following local density [9]

$$\rho(x_1, \dots, x_N) = \left| \det_{1 \leq i, j \leq N} (s_i(x_j))_{i,j} \right|^2 e^{-\psi(x_1)} \dots e^{-\psi(x_N)} / Z_\psi$$

where  $(s_i)$  is an orthonormal base in the Hilbert space  $(H^0(X, L+K_X), \langle \cdot, \cdot \rangle_{\psi_0})$  and  $Z_\psi$  is the normalizing constant (partition function). Integration over  $X^N$  will be denoted by respect to  $\gamma$  by  $\mathbb{E}_\psi$  (=expectation). As is essentially well-known [9]  $\log Z_\psi = -N\mathcal{L}_{\omega_0}(u)$ , i.e.

$$(5.4) \quad \mathcal{L}_{\omega_0}(u) = -\log \mathbb{E}_{\psi_0}(e^{-(u(x_1)+\dots+u(x_n))})/N$$

We also have [9]

$$(5.5) \quad \beta_u(x) = \frac{1}{N} \mathbb{E}_\psi \left( \sum_{i=1}^N \delta_{x_i} \right) = \int_{X^{N-1}} |(\det S_0)(x, x_2, \dots, x_N)|^2 e^{-\psi(x)} e^{-\psi(x_2)} \dots e^{-\psi(x_N)} / Z_\psi$$

In particular, the map  $(x, t) \mapsto (\beta_{u_t}(x)/\omega_0^n)$  is *continuous* if  $u_t$  is a continuous path and hence there is a positive constant  $C$  such that

$$(5.6) \quad 1/C \leq (\beta_{u_t}(x)/\omega_0^n) \leq C$$

on  $[0, 1] \times X$ , if  $L + K_X$  is globally generated, i.e. if  $\beta_{u_t}(x) > 0$  point-wise. Formula 5.5 also shows, by the dominated convergence theorem, that  $\frac{d\beta_{u_t}(x)}{dt} \big|_{t=0+}$  exists under the assumptions in the following lemma.

**Lemma 38.** *Let  $u_t$  be a family of continuous functions on  $X$  such that the right derivative  $v_t := \frac{du_t}{dt} \big|_{t=0+}$  exists and is uniformly bounded on  $[0, 1] \times X$ .*

Then

$$(5.7) \quad = \frac{d\beta_{u_t}(x)}{dt} \Big|_{t=0+} = \int_{X_y} |K_u(x, y)|^2 e^{-(\psi_0(x)+\psi_0(y))} v_0(y) - \beta_{u_t}(x) v_0(x) := R[v](x)$$

*Proof.* The proof of the formula was obtained in [14] (formula 5), at least in the smooth case. For completeness we recall the simple proof. By the discussion above we may differentiate formula 5.3 and use Leibniz product rule to get

$$\partial_t(K_t(x, x)) = 2\operatorname{Re} \int_{X_y} \partial_t(K_t(x, y)) \wedge \bar{K}(x, y) e^{-\psi_t(y)} - \int_{X_y} |K_t(x, y)|^2 (\partial_t \psi_t(y)) e^{-(\psi_t(x)+\psi_t(y))}$$

Applying formula 5.1 to the holomorphic section  $s(\cdot) = \partial_t K_t(x, \cdot)$  shows that the second term above equals  $2\partial_t(K_t(x, x))$ . Hence,

$$\partial_t(K_t(x, x)) = \int_{X_y} |K_t(x, y)|^2 (\partial_t u_t(y)) e^{-(\psi_t(x)+\psi_t(y))},$$

which proves the lemma, since  $N\beta_u(x) = K(x, x)e^{-\psi(x)}$ .  $\square$

**5.2. A “Bergman kernel proof” of the convexity statement in Theorem 12.** Let  $\psi_t := \psi_0 + u_t$ . As will be shown below, differentiating  $\mathcal{L}_{\omega_0}(u_t)$  gives

$$(5.8) \quad \partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(u_t) = \frac{1}{N} \sum_{i=1}^N (\|(\partial_t \partial_{\bar{t}} u_t) s_i\|_{\psi_t}^2 - \|(\partial_{\bar{t}} u_t s_i) - \Pi_{u_t}(\partial_{\bar{t}} u_t s_i)\|_{\psi_t}^2),$$

where  $(s_i)$  is orthonormal wrt  $\psi_t = \psi_0 + u_t$ . Given this formula the argument proceeds exactly as in [15]; by the definition of  $\Pi_{u_t}$ , the second term inside the sum is the  $L^2$ -norm of the smooth solution  $\alpha$  to the inhomogenous  $\bar{\partial}$ -equation on  $X$ :

$$(5.9) \quad \bar{\partial}_X \alpha = \bar{\partial}_X(\partial_t u_t) s_i,$$

which has minimal norm wrt  $\|\cdot\|_{\psi_t}^2$ . Now the Hörmander-Kodaira  $L^2$ -inequality for the solution gives

$$(5.10) \quad i^{n^2} \int_X \alpha \wedge \bar{\alpha} e^{-\psi_t} \leq i^{n^2} \int_X |\bar{\partial}_X(\partial_t u_t)|_{\omega_{u_t}}^2 s \wedge \bar{s} e^{-\psi_t},$$

using that  $\omega_{u_t} > 0$ . Hence, by formula 5.8,

$$(5.11) \quad \partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(u_t) \geq \frac{1}{N} \sum_{i=1}^N (\|(\partial_t \partial_{\bar{t}} u_t) - |\bar{\partial}_X(\partial_t u_t)|_{\omega_{u_t}}^2 s_i\|_{\psi_t}^2)$$

But since, by assumption,  $(dd^c U + \pi_X^* \omega_0)^{n+1} \geq 0$  the rhs is non-negative (compare formula 3.25), which proves that  $\mathcal{L}_{\omega_0}(u_t)$  is *convex* wrt real  $t$ . Note that  $\mathcal{L}_{\omega_0}(u_t)$  is *affine* precisely when 5.10 is an *equality*. By examining the Bochner-Kodaira-Nakano-Hörmander *identity* implying the inequality 5.10 one sees that the remaining term appearing in the identity has to vanish. In turn, this is used to show that the vector field  $V_t$

defined by formula 2.6 has to be *holomorphic* on  $X$  (see [15]). Integrating  $V_t$  finally gives the existence of the automorphism  $S_1$  in Theorem 12, as explained in section 3.1. A slight generalization of this argument, in the one dimensional case, will be given in the next section.

In [15] formula 5.8 was derived using the general formalism of holomorphic vector bundles and their curvature. We will next give an alternative “Bergman kernel proof”. First formula 5.7 and Leibniz product rule give

$$\partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(u_t) = \int_X (\partial_t \partial_{\bar{t}} u_t) \beta_{u_t} + \frac{d\beta_{u_t}}{dt} \Big|_{t=0+} (\partial_t u_t)$$

Next, by formula 5.7 the second term may be expressed in terms of the Bergman kernel  $K_t(x, y)$  associated to the weight  $\psi_t$  as

$$\frac{1}{N} \int_{X \times X} |K_t(x, y)|^2 e^{-(\psi_t(x) + \psi_t(y))} ((\partial_t u_t)(x)(\partial_t u_t)(y) - \int_X \beta(\partial_t u_t)^2,$$

By simple and well-known identities for Toeplitz operators this last expression, for  $t = 0$ , is precisely the trace of the operator  $T[\partial_t u_t]^2 - T[(\partial_t u_t)^2]$ . All in all we obtain

$$\partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(u_t) = \frac{1}{N} \text{Tr}(T[\partial_t \partial_{\bar{t}} u_t] + (T[\partial_t u_t])^2 - T[(\partial_t u_t)^2]),$$

for  $t = 0$ . Expanding in terms of an orthonormal base  $s_i$  hence gives

$$\partial_t \partial_{\bar{t}} \mathcal{L}_{\omega_0}(u_t) = \frac{1}{N} \sum_{i=1}^N (\|(\partial_t \partial_{\bar{t}} u_t) s_i\|_{\psi_0 + u_t}^2 + \|\Pi_{u_t}(\partial_t u_t s_i)\|_{\psi_0 + u_t}^2 - \|\partial_t u_t s_i\|_{\psi_0 + u_t}^2),$$

for  $t = 0$  (and hence for all  $t$  by symmetry) which finally proves 5.8, using “Pythagora’s theorem”.

**5.3. Conditions for equality in the Hörmander-Kodaira estimate when  $n = 1$ .** Denote by  $\Omega^{p,q}(X, L) = \{f^{p,q}\}$  the space of all smooth  $(p, q)$ -forms with values in the holomorphic line bundle  $L \rightarrow X$ . We will next assume that  $\dim X = 1$  and use the natural identification  $\Omega^{1,q}(X, L) = \Omega^{0,q}(X, L + K_X)$ . Given a metric on  $L$  that we will write, abusing notation slightly, as  $e^{-\psi}$  we get a natural Hermitian product on  $\Omega^{1,0}(X, L)$ . The next proposition is a generalization of Proposition 2.2. in [15] to possible degenerate curvature forms  $i\partial\bar{\partial}\psi \geq 0$ , in the one dimensional case.

**Proposition 39.** *Let  $L$  be a line bundle over a compact complex manifold  $X$  of dimension one and let  $e^{-\psi}$  be a smooth metric on  $L$  such that  $i\partial\bar{\partial}\psi \geq 0$  with strict inequality almost everywhere. For any given  $g^{1,1} \in \Omega^{1,1}(X, L)$  the  $L^2(e^{-\psi})$ -minimal solution of*

$$\bar{\partial}\alpha^{1,0} = g^{1,1}$$

*satisfies*

$$\|\alpha^{1,0}\|^2 := i \int_X \alpha^{1,0} \wedge \overline{\alpha^{1,0}} e^{-\psi} \leq \|g^{1,1}\|^2 := i \int_X g^{1,1} \frac{\overline{g^{1,1}}}{\partial\bar{\partial}\psi} e^{-\psi} < \infty$$



with equality if and only if  $g^{1,1}/\partial\bar{\partial}\psi$  defines a holomorphic section of  $L$  over  $X$ .

Next, we turn to the proof of the proposition, which is rather close to that in [15], but taking advantage of the specially simple structure of one complex dimension. As a courtesy to the reader it will be essentially self-contained. Let  $D$  denote the Chern connection on the Hermitian holomorphic line bundle  $(L, e^{-\psi})$  and decompose  $D = D^{1,0} + \bar{\partial}$  according to bidegree. In other words, the operator  $D^{1,0}$  on  $\Omega^{0,0}(X, L)$  is uniquely determined by the relation

$$\langle D^{1,0}f^{0,0}, g^{1,0} \rangle := i \int_X D^{1,0}f^{0,0} \wedge \overline{g^{0,1}}e^{-\psi} = -i \int_X f^{0,0} \overline{\partial g^{0,1}}e^{-\psi}$$

for all  $f^{0,0}$  and  $g^{0,1}$ . Note that in the usual local representations where the metric in  $L$  is represented by  $e^{-\psi}$  this means that  $D^{1,0} = \partial - \partial\psi \wedge$  locally, which gives the following basic commutation relation:

$$[\bar{\partial}, D^{1,0}] = \partial\bar{\partial}\psi,$$

where  $\partial\bar{\partial}\psi$  is the (non-normalized) curvature form. Now, for any given smooth  $g^{1,1}$  we consider the following equation for  $f^{0,0}$  smooth

$$(5.12) \quad \bar{\partial}D^{1,0}f^{0,0} = g^{1,1},$$

Note that, if  $f^{0,0}$  is a solution then, by 5.3,  $D^{1,0}f^{0,0}$  is orthogonal to the null space of  $\bar{\partial}$  on  $\Omega^{1,0}(X, L)$  and hence  $u^{1,0} := D^{1,0}f^{0,0}$  is the  $L^2$ -minimal solution of the inhomogenous  $\bar{\partial}$ -equation in Prop 39. Next, we observe that the equation 5.12 admits a smooth solution, which is unique. Indeed, since  $\bar{\partial}D^{1,0}$  is elliptic this is equivalent to the null space of  $\bar{\partial}D^{1,0}$  being trivial, i.e. that the null space of  $D^{1,0}$  is trivial (using 5.3). This latter fact follows in turn from the following Hörmander-Kodaira identity obtained by integrating the commutation relation above against  $|f^{0,0}|^2$  and using 5.3 again: for any  $f^{0,0} \in \Omega^{0,0}(X, L)$  :

$$(5.13) \quad \|D^{1,0}f^{0,0}\|^2 + \|\bar{\partial}f^{0,0}\|^2 = \|f^{0,0}\|^2 := i \int_X |f^{0,0}|^2 \partial\bar{\partial}\psi,$$

where the last equality defines an Hermitian product on  $\Omega^{0,0}(X, L)$ . In particular, if  $D^{1,0}f^{0,0} = 0$  on  $X$  then  $f^{0,0}$  vanishes on an open set, which by the identity principle for homogenous elliptic equations means that it vanishes identically, which finishes the proof of the existence and uniqueness.

Next, we define the following map, introducing the (possibly degenerate) Hermitian product on the image that makes the map into a unitary one:

$$* : \Omega^{0,0}(X, L) \rightarrow \Omega^{1,1}(X, L), \quad *f^{0,0} := if^{0,0}\partial\bar{\partial}\psi$$

whose inverse, defined on the image  $*(\Omega^{0,0}(X, L))$  will also be denoted by  $*$ . Then

(5.14)

$$\|D^{1,0}f^{0,0}\|^2 = i \int_X (\bar{\partial} D^{1,0}f^{0,0}) \overline{f^{0,0}} \leq \|\bar{\partial} D^{1,0}f^{0,0}\| \|*f^{0,0}\| \leq \|\bar{\partial} D^{1,0}f^{0,0}\| \|D^{1,0}f^{0,0}\|$$

(using the Cauchy-Schwartz inequality in the first inequality and 5.13 in the second one), which proves the estimate in the proposition above upon division by  $\|D^{1,0}f\|$ . Next, if equality holds in the estimate in the proposition above then this forces equalities in 5.14. Hence, the first equality gives equality in the Cauchy-Schwartz inequality and hence  $*f^{0,0} = \bar{\partial} D^{1,0}f^{0,0} (= g^{1,1})$  (possibly up to a multiplicative constant of unit modulus). Concretely, this means that  $g^{1,1}/i\partial\bar{\partial}\psi = f^{0,0}$ , which is smooth on all of  $X$  by ellipticity. But then the second equality above gives  $\|*f\|^2 = \|D^{1,0}f\|^2$  so that the identity 5.13 forces  $\bar{\partial}f^{0,0} = 0$ , i.e.  $\bar{\partial}(*g) = 0$ . Conversely, if  $\bar{\partial}(*g) = 0$  then the equality in the estimate follows immediately from the identity 5.13.

**Corollary 40.** *Let  $L$  be a line bundle over a compact complex manifold  $X$  of dimension one and let  $e^{-\psi_t}$  be a one-parameter family of metrics on  $L$  such that*

- *for each  $t$   $\partial_t\partial_t\psi_t$  is in  $L^1(X, \beta_{\psi_t})$ , where  $\beta_{\psi_t}$  is the Bergman measure*
- *for each  $t$   $\psi_t$  is smooth on  $X$  with  $i\partial_X\bar{\partial}_X\psi_t$  strictly positive almost everywhere and*
- *for all  $h^{1,0} \in H^{1,0}(X, L)$  we have*

$$ih^{1,0} \wedge \overline{h^{1,0}} e^{-\psi_t} \leq C_t i\partial\bar{\partial}\psi_t$$

*on  $X$  for some constant  $C_t$  depending on  $t$ .*

Then,  $d^2\mathcal{L}(\psi_t)/d^2t = 0$  at  $t$  (see definition 1.5) if and only if there is  $h^{1,0} \in H^{1,0}(X, L)$  such that  $h^{1,0} \wedge \bar{\partial}(\partial_t\psi_t)/\partial_X\bar{\partial}_X\psi_t$  defines a holomorphic section of  $L$  over  $X$ .

*Proof.* Under the assumptions above the proof in the previous section carries over essentially verbatim to give equality in 5.11. In particular, we can take  $h^{1,0} = s_i$  for some  $i$  and apply the previous proposition to  $g^{1,1} := h^{1,0} \wedge \bar{\partial}(\partial_t\psi_t)$  which, by assumption, satisfies  $\|g^{1,1}\|^2 := i \int_X g^{1,1} \overline{g^{1,1}} e^{-\psi} / \partial\bar{\partial}\psi < \infty$ .  $\square$

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